MATHEMATICS MAGAZINE

ARTICLES BY TITLES

INDEX TO VOLUME 37 JANUARY THROUGH DECEMBER, 1964

ARTICLES BY AUTHORS

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

PROPOSALS

SOLUTIONS

Quickies and Answers

The page on which Quickies, Trickies and Falsies appear is in the parentheses following the number of these problems; the page on which the Answers, Solutions, and Explanations appear is in boldface.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Names of authors are in ordinary type, those of reviewers in capitals and small capitals

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CONTENTS

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A GEOMETRICAL APPROACH TO PROBABILITY

H. D. BRUNK AND L. G. GREF, University of California, Riverside

1. Introduction. For the development of various aspects of the theory of probability, it has been found convenient to think of random variables as elements of a Hilbert space, particularly in connection with linear methods (e.g., [4], Ch. 4, [5], Ch. 10).

The authors believe that an approach to the theory of probability making explicit appeal to geometric intuition in discussing properties of random variables offers distinct advantages. The concept of *expectation* of a random variable, otherwise somewhat difficult to motivate, is interpreted in terms of "the closest constant random variable to the given random variable," and *probability* is defined as an expectation; *variance* is proportional to the square of the minimum distance. *Conditional expectation* is the result of an orthogonal projection.

The first published mention of conditional expectation as projection known to the authors appears as a footnote in [2]. A number of studies of conditional expectation from this point of view have been made ([6], [1], [8], [3]). The desirability of treating random variables as elements of a Hilbert space was pointed out to one of the authors by I. E. Segal, who adopted this point of view in [7].

In order to keep the development as simple and clear as possible, only finite probability spaces are considered here. However, the methods and concepts of the theory of inner product spaces on which this approach is based carry over largely to general probability spaces.

2. Random variables on a finite probability space. A finite probability space is a finite set Ω: $\{\omega_1, \omega_2, \cdots, \omega_n\}$ of elements, called *elementary events*, together with corresponding relative "odds" or weights: positive numbers, $m_1, m_2, \cdots, m_n$. The elementary events represent possible results or outcomes of an experiment to be performed. The weights are interpreted as odds: if $m_2 > m_1$, the odds in favor of ω_2 over ω_1 are m_2 to m_1; thus if $m_2 = 2m_1$, ω_2 is twice as likely to occur as ω_1, etc.

A *random variable* X is a real valued function on Ω: X associates with each elementary event ω_i a real number x_i, the value, $X(\omega_i)$, of X at ω_i, $i = 1, 2, \cdots, n$. A random variable X on Ω will be represented as an n-dimensional vector, its ith component being the value of X at ω_i, $i = 1, 2, \cdots, n$: $X = (x_1, x_2, \cdots, x_n)$.

Example 2.1. A balanced die is to be tossed. There are six elementary events, one corresponding to each possible fall of the die. Let ω_i denote the elementary event corresponding to the face with i spots, $i = 1, 2, \cdots, 6$. The random variable X described verbally as "the number of spots which will show on the upturned face" has the value 1 at the elementary event ω_1, the value 2 at ω_2, etc., and is represented by the vector (1, 2, 3, 4, 5, 6). The random variable Y: "the excess of the number of spots over 3" is represented by $(-2, -1, 0, 1, 2, 3)$.

Example 2.2. A thumbtack is to be tossed twice. There are four elementary events: ω_1: "the point will fall up both times," which might also be represented as {up, up}; $\omega_2=$ {up, down}, "the point will fall up the first time and down the second"; $\omega_3=$ {down, up}, $\omega_4=$ {down, down}. The odds m_1, m_2, m_3, m_4 might be estimated using the frequencies observed in a long series of tosses of the thumbtack. The random variable X described verbally as: "the number of times the tack will fall point up" will take the value 2 if $\omega_1=$ {up, up} occurs, the value 1 if either ω_2 or ω_3 occurs, and the value 0 if ω_4 occurs, and is represented as the vector (2, 1, 1, 0).

DEFINITIONS 2.1.

(i) A *constant* random variable has equal components: $C=(c, \cdots, c)$, where c is a real number. That is, $C(\omega_i)=c$, $i=1, 2, \cdots, n$. In particular, the unit random variable is $U=(1, \cdots, 1)$.

(ii) An *event* is a set of elementary events. The *indicator random variable* I_A of an event A is the random variable taking the value 1 at each elementary event in A (favorable to A), and otherwise taking the value 0. (This is the characteristic function of the set A.) In Example 2.1, the elementary events ω_2, ω_4 and ω_6 comprise the event: "an even number of spots will show," and its indicator random variable is (0, 1, 0, 1, 0, 1). In Example 2.2, the random variable (1, 1, 0, 0) indicates the event: "the thumbtack will fall point up on the first toss."

In the following definitions, X, Y, and Z represent random variables $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$, $(z_1, z_2, \cdots, z_n)$ respectively.

(iii) The *sum* of two random variables is given by

$$X + Y \equiv_D (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n).$$

(iv) Multiplication of a random variable by a scalar is defined as follows:

$$kX \equiv_D (kx_1, kx_2, \cdots, kx_n),$$

where k is a real number. For example, the constant random variable C above may be written: $C=cU$.

(v) We define the *product* of two random variables X and Y by

$$XY \equiv_D (x_1y_1, x_2y_2, \cdots, x_ny_n).$$

(vi) A *function* $\phi(x)$ of a random variable X is defined by

$$\phi(X) \equiv_D (\phi(x_1), \phi(x_2), \cdots, \phi(x_n)),$$

where ϕ denotes a function from the real numbers into the reals.

(vii) The inner product of two random variables is defined to be

$$(X, Y) \equiv_D \sum_{i=1}^{n} m_i x_i y_i,$$

where m_i is the weight associated with the elementary event ω_i $(i=1, 2, \cdots, n)$.

It is immediate that the inner product thus defined has the properties usually required of an inner product:

$$\text{(a)}\quad (X + Y, Z) = (X, Z) + (Y, Z),$$

(b) $(kX, Y) = k(X, Y)$ for each real number k,

(c) $(X, Y) = (Y, X)$,

(d) $(X, X) > 0$ if $X \neq 0$.

We have required that the weights $m_1, \cdots, m_n$ be positive, and thus (d) above holds. If one or more of the weights were zero, a modification of the concept "random variable" would be required in order that (d) might hold. One would introduce an equivalence relation on the class of vectors, two vectors being equivalent if all components associated with nonzero weights are equal. A random variable would then be an equivalence class of vectors. An analogous definition of random variable is required for general (not necessarily finite) probability spaces.

The inner product defined above permits the introduction of a metric, rendering the space of random variables an n-dimensional euclidean space:

(viii) the *magnitude* of a random variable is given by

$$\|X\| \equiv_D (X, X)^{1/2};$$

(ix) we define the *distance* between two random variables as

$$d(X, Y) \equiv_D \|X - Y\|.$$

(That the distance function d defines a metric is not difficult to show; i.e., $d(X, Y) \geqq 0$ and equality holds if and only if $X = Y$, $d(X, Y) = d(Y, X)$, and $d(X, Y) + d(Y, Z) \geqq d(X, Z)$).

In the case of equal odds, $m_1 = m_2 = m_3 = 1$, with $n = 3$, we have the familiar geometrical interpretation of inner product: $(X, Y) = \|X\| \, \|Y\| \cos\theta$, where θ is the angle between the vectors X and Y. This same interpretation can be used for arbitrary n with weights m_i $(i = 1, 2, \cdots, n)$ to define an "angle" between random variables. Two random variables X and Y are defined to be *orthogonal* or *perpendicular*, $X \perp Y$, if and only if $(X, Y) = 0$.

The set of all random variables over a given finite space is a real vector space with the operations defined above. The set of constant random variables is a subspace: a subset containing all linear combinations of its elements.

A special property of the particular inner product defined above is:

$$(XY, Z) = (X, YZ).$$

3. Expectation, variance, and probability.

THEOREM 3.1. *Given a random variable X, the closest constant random variable, C, to X, in the sense of minimizing the distance $\|X - C\|$, is $C = (c, c, \cdots, c) = cU$, where*

$$c = (X, U)/(U, U) = \sum_{i=1}^{n} m_i x_i \Big/ \sum_{i=1}^{n} m_i.$$

This number $c = (X, U)/(U, U) = E(X)$ is defined to be the *expectation* of the random variable X. The closest constant random variable C is also called the *projection* of X on the subspace of constant random variables. The quantity $\|X-C\|^2\sum_{i=1}^n m_i = V(X)$ is the *variance* of X.

Proof of Theorem 3.1. Since $\|X-C\|$ is nonnegative, minimizing $\|X-C\|$ is equivalent to minimizing $\|X-C\|^2$. Using the above properties of inner product we have

$$\begin{aligned}\|X - C\|^2 &= (X - C, X - C)\\ &= (X - cU, X - cU)\\ &= (X, X) - 2c(X, U) + c^2(U, U)\\ &= (X, X) + (U, U)[c - (X, U)/(U, U)]^2\\ &\qquad\qquad - (X, U)^2/(U, U).\end{aligned}$$

Since the second term is nonnegative, the minimum is achieved for

$$c = (X, U)/(U, U) = \sum_{i=1}^n m_i x_i \Big/ \sum_{i=1}^n m_i.$$

For some applications involving infinite sets of elementary events it may be important to leave the weights $\{m_i\}$ unnormalized; this is the reason we have not heretofore assumed the weights chosen so that $\sum_{i=1}^n m_i = 1$. However, for finite sets of elementary events there is no loss of generality in assuming that each of the original odds has been divided by their sum so that $\sum_{i=1}^n m_i = (U, U) = 1$ and, henceforth, we make this simplifying assumption. Thus

$$E(X) = (X, U) = \sum_{i=1}^n m_i x_i$$

and

$$V(X) = \|X - C\|^2 = (X, X) - (X, U)^2 = \sum_{i=1}^n m_i x_i^2 - \left(\sum_{i=1}^n m_i x_i\right)^2.$$

Since $(X, X) = (X, XU) = (X^2, U) = E(X^2)$, $V(X)$ may also be written as $V(X) = E(X^2) - [E(X)]^2$.

If I_A is the indicator random variable of an event A, we define the probability of A by $P(A) = E(I_A) = \sum_{\omega_i \in A} m_i$.

DEFINITIONS 3.1.

(i) Let Y and Z be indicator random variables of events A and B, respectively. The *union* of the events A and B, $A \cup B$, can be represented by the indicator random variable $Y \vee Z = (y_1 \vee z_1, y_2 \vee z_2, \cdots, y_n \vee z_n)$ where $y_i \vee z_i = \max(y_i, z_i)$ $(i = 1, 2, \cdots, n)$.

For example, if $Y = (1, 1, 0, 0, 1, 1)$ and $Z = (1, 0, 1, 0, 1, 0)$ then $Y \vee Z = (1, 1, 1, 0, 1, 1)$.

(ii) If Y and Z are as in definition (i) above, then the *intersection* of the events A and B, written $A \cap B$ or AB, can be represented by the indicator random variable $Y \wedge Z = (y_1 \wedge z_1, y_2 \wedge z_2, \cdots, y_n \wedge z_n)$ where $y_i \wedge z_i = \min(y_i, z_i)$ $(i = 1, 2, \cdots, n)$. (In the example above, $Y \wedge Z = (1, 0, 0, 0, 1, 0)$.)

(iii) Two events A and B are said to be *mutually exclusive* if the indicator $Y \wedge Z$ of $A \cap B$ is 0; thus two events are mutually exclusive if and only if their indicator random variables are perpendicular or orthogonal.

THEOREM 3.2. *If X and Y are random variables and k a real number then $E(X+Y)=E(X)+E(Y)$ and $E(kX)=kE(X)$.*

Proof. This is easily seen to follow from the definition of expectation and the bilinearity of the inner product.

COROLLARY 3.1. *For events A and B,*

$$P(A \cup B) + P(A \cap B) = P(A) + P(B).$$

Proof. Let Y and Z be the indicator random variable of the events A and B. Then $Y \vee Z$ is the indicator of the event $A \cup B$, and $Y \wedge Z$ is the indicator of $A \cap B$;

$$\begin{aligned} P(A \cup B) + P(A \cap B) &= E(Y \vee Z) + E(Y \wedge Z) \\ &= E(Y \vee Z + Y \wedge Z). \end{aligned}$$

But, $Y \vee Z + Y \wedge Z = Y + Z$, so that

$$\begin{aligned} P(A \cup B) + P(A \cap B) &= E(Y + Z) \\ &= E(Y) + E(Z) = P(A) + P(B). \end{aligned}$$

COROLLARY 3.2. *If A and B are mutually exclusive events, then $P(A \cup B) = P(A)+P(B)$.*

THEOREM 3.3. *For an arbitrary function ϕ, $E[\phi(X)] = \sum \phi(x) \Pr\{X=x\}$.* (The summation is extended over all values x of the random variable X; $\Pr\{X=x\}$ refers to the probability of the event $\{X=x\}$ consisting of all elementary events ω_i at which X takes the value x.)

Proof. We have $\phi(X) = \sum_x \phi(x) I_{\{x\}}$ where $I_{\{x\}}$ is the indicator random variable of the event $\{X=x\}$. Hence $E[\phi(X)] = \sum_x \phi(x) E(I_{\{x\}}) = \sum_x \phi(x) \Pr\{X=x\}$.

COROLLARY 3.3. $E(X) = \sum_x x \Pr\{X=x\}$.

4. Conditional expectation, independence, and linear regression. Before proceeding with this section, we review certain elementary properties of finite-dimensional euclidean spaces. We say that a set of vectors $\{e_1, \cdots, e_r\}$ *generates* a vector space or subspace $\mathfrak{U}$ if every vector Y in $\mathfrak{U}$ is a linear combination of the vectors $\{e_i\}$; that is $Y = \sum_{i=1}^{n} k_i e_i$. If in addition to generating the space $\mathfrak{U}$ the $\{e_i\}$ satisfy the additional conditions: (a) $e_i \perp e_j$ for $i \neq j$ and (b) $\|e_i\| = 1$ for $i = 1, 2, \cdots, r$, we say that the $\{e_i, \cdots, e_r\}$ forms an *orthonormal basis* for $\mathfrak{U}$.

Throughout this section, attention is restricted to spaces $\mathfrak{V}$ of random variables on finite probability spaces.

THEOREM 4.1. *If $\mathfrak{U}$ is a finite dimensional subspace of $\mathfrak{V}$, and if $X \in \mathfrak{V}$, $\|X - Y\|$ considered as a function of Y is minimized by precisely one element Y in $\mathfrak{U}$.*

Proof. Let $\{e_1, \cdots, e_r\}$ form an orthonormal basis for $\mathcal{U}$. Then for every $Y \in \mathcal{U}$ we have $Y = \sum_{i=1}^{r} k_i e_i$ for certain real numbers k_i, $i = 1, 2, \cdots, r$. We have

$$\begin{aligned}
\|X - Y\|^2 &= (X - Y, X - Y) \\
&= (X, X) - 2(X, Y) + (Y, Y) \\
&= (X, X) - 2\left(X, \sum_{i=1}^{r} k_i e_i\right) + \left(\sum_{i=1}^{n} k_i e_i, \sum_{j=1}^{n} k_j e_j\right) \\
&= (X, X) - 2\sum_{i=1}^{r} k_i (X, e_i) + \sum_{i=1}^{r} k_i^2 \\
&= (X, X) - \sum_{i=1}^{r} (X, e_i)^2 + \sum_{i=1}^{r} \left(k_i^2 - 2k_i(X, e_i) + (X, e_i)^2\right) \\
&= (X, X) - \sum_{i=1}^{r} (X, e_i)^2 + \sum_{i=1}^{r} (k_i - (X, e_i))^2.
\end{aligned}$$

Since the last term is nonnegative, the minimum is achieved for $k_i = (X, e_i)$, $(i = 1, 2, \cdots, r)$ and for these numbers $\{k_i\}$ alone. Thus $Y = \sum_{i=1}^{r} (X, e_i) e_i$ is the unique element of $\mathcal{U}$ minimizing $\|X - Y\|$ in $\mathcal{U}$.

This closest point of $\mathcal{U}$ to X is termed the *orthogonal projection* of X on $\mathcal{U}$, and will be denoted by $E(X | \mathcal{U})$; it might also be called the conditional expectation of X given $\mathcal{U}$. The former term is justified in Theorem 4.2.

THEOREM 4.2. *If $\mathcal{U}$ is a subspace of $\mathcal{V}$ and X is a fixed element of $\mathcal{V}$ then Y is the element of $\mathcal{U}$ minimizing $\|X - Y\|$ in $\mathcal{U}$ if and only if $X - Y$ is orthogonal to every vector Z in $\mathcal{U}$.*

Proof. Again we assume a finite orthonormal basis for $\mathcal{U}$, $\{e_1, e_2, \cdots, e_r\}$. We have $Y = \sum_{i=1}^{r} (X, e_i) e_i$ from Theorem 4.1. Let $Z = \sum_{i=1}^{r} k_i e_i$ be an arbitrary element of $\mathcal{U}$; then

$$\begin{aligned}
(X - Y, Z) &= (X, Z) - (Y, Z) \\
&= \left(X, \sum_{i=1}^{r} k_i e_i\right) - \left(\sum_{i=1}^{r} (X, e_i) e_i\right) \sum_{j=1}^{r} k_j e_j \\
&= \sum_{i=1}^{r} k_i (X, e_i) - \sum_{i=1}^{r} \sum_{j=1}^{r} ((X, e_i) e_i, k_j e_j) \\
&= \sum_{i=1}^{r} k_i (X, e_i) - \sum_{i=1}^{r} \sum_{j=1}^{r} (X, e_i) k_j (e_i, e_j) \\
&= \sum_{i=1}^{r} k_i (X, e_i) - \sum_{i=1}^{r} (X, e_i) k_i \\
&= 0.
\end{aligned}$$

Therefore $X - Y$ and Z are orthogonal; that is $[X - E(X | \mathcal{U})] \perp Z$ for every $Z \in \mathcal{U}$.

THEOREM 4.3. *If a and b are real numbers and $\mathcal{U}$ is a subspace of $\mathcal{V}$, then* $E(aX+bY|\mathcal{U})=aE(X|\mathcal{U})+bE(Y|\mathcal{U})$.

Proof. Let $X^*=E(X|\mathcal{U})$, $Y^*=E(Y|\mathcal{U})$. If $Z\in\mathcal{U}$, then $(aX+bY-aX^*-bY^*, Z)=a(X-X^*, Z)+b(Y-Y^*, Z)=0$. From Theorem 4.2 it then follows that $aX^*+bY^*=E(aX+bY|\mathcal{U})$.

THEOREM 4.4. *If $\mathcal{U}$ is a subspace of $\mathcal{V}$, and if $XZ\in\mathcal{U}$ for every $Z\in\mathcal{U}$, then* $E(XY|\mathcal{U})=XE(Y|\mathcal{U})$.

Proof. Let $Y^*=E(Y|\mathcal{U})$. If $Z\in\mathcal{U}$, then $(XY-XY^*, Z)=(Y-Y^*, XZ)=0$.

The conclusion of the theorem then follows from Theorem 4.2.

THEOREM 4.5. *If $\mathcal{W}$ is a subspace of $\mathcal{U}$, which in turn is a subspace of $\mathcal{V}$, then $E(X|\mathcal{W})=E\{E(X|\mathcal{U})|\mathcal{W}\}$. That is, the projection of X on $\mathcal{W}$ coincides with the projection on $\mathcal{W}$ of the projection of X on $\mathcal{U}$.*

Proof. Let $Y_{\mathcal{U}}=E(X|\mathcal{U})$ so that $(X-Y_{\mathcal{U}})\perp Z$ for all $Z\in\mathcal{U}$; in particular, $(X-Y_{\mathcal{U}})\perp R$ for all $R\in\mathcal{W}$, so that $(X-Y_{\mathcal{U}}, R)=0$ for all R in $\mathcal{W}$. If $Y_{\mathcal{W}}=E(Y_{\mathcal{U}}|\mathcal{W})$, then $(Y_{\mathcal{U}}-Y_{\mathcal{W}}, R)=0$ for all R in $\mathcal{W}$, and addition yields $(X-Y_{\mathcal{W}}, R)=0$ for all R in $\mathcal{W}$. By Theorem 4.2, $Y_{\mathcal{W}}$ is $E(X|\mathcal{W})$, completing the proof of the theorem.

DEFINITION 4.1. Given a random variable $Z=(z_1, \cdots, z_n)$, let $\mathcal{V}_Z$ denote the set of all random variables which are functions of Z: $W=(w_1, \cdots, w_n)\in\mathcal{V}_Z$ if and only if there is a function f such that $w_i=f(z_i)$, $i=1, 2, \cdots, n$. The set $\mathcal{V}_Z$ of random variables is a subspace of the vector space of all random variables, and if X is a random variable there is, by Theorem 4.1, a closest random variable Y in $\mathcal{V}_Z$ to X. This random variable Y is the *conditional expectation* of X given Z, denoted by $E(X|Z)$. If $\{e_1, \cdots, e_r\}$ is an orthonormal basis for $\mathcal{V}_Z$, then $E(X|Z)=\sum_{i=1}^{r}(X, e_i)e_i$.

THEOREM 4.6. *For $i=1, 2, \cdots, n$, let $T_i=\{j: Z_j=z_i\}$, and set $y_i=\sum_{j\in T_i}m_jx_j/\sum_{j\in T_i}m_j$. Then $E(X)|Z=(y_1, \cdots, y_n)$.*

Proof. Let $c_1, c_2, \cdots, c_r$ be the distinct values of Z, and set $A_k=\{i: z_i=c_k\}$, $k=1, 2, \cdots, r$. Let $Y=g(Z)=E(X|Z)$. If f is an arbitrary function, let $f_k=f(c_k)$. Since $X-Y\perp f(Z)$ for all f, we have

$$\sum_{k=1}^{r} f_k \sum_{i\in A_k} [x_i - g(c_k)]m_i = 0,$$

for all real numbers $f_1, f_2, \cdots, f_r$. Therefore,

$$\sum_{i\in A_k} [x_i - g(c_k)]m_i = 0, \qquad k = 1, 2, \cdots, r,$$

so that

$$g(c_k) = \sum_{i\in A_k} x_im_i \Big/ \sum_{i\in A_k} m_i, \qquad k = 1, 2, \cdots, r,$$

and, by definition of A_k and $g(c_k)$, we have the desired result. Note that if $E(X|Z=z)$ is defined to be $g(z)$, with $\Pr[Z=z]>0$ and $g(Z)=E(X|Z)$, we have

$$E(X|Z=z) = \sum_{z_j=z} m_j x_j \Big/ \sum_{z_j=z} m_j.$$

We note in passing that given a random variable $Z=\{z_1, \cdots, z_n\}$ one can easily construct an orthonormal basis for $\mathcal{V}_Z$ as follows. For each possible value z of Z, let I_z denote the indicator random variable of the event $\{Z=z\}$; $E(I_z)$ is the probability of this event. The random variables $I_z/[E(I_z)]^{1/2}$ are readily seen to form an orthonormal basis for $\mathcal{V}_Z$. In particular, if Z is the indicator random variable of an event B (not the sure event nor the impossible event) so that $\bar{Z}=U-Z$ is the indicator random variable of its complement $\bar{B}$, an orthonormal basis for $\mathcal{V}_Z$ is furnished by $Z/[P(B)]^{1/2}$ and $\bar{Z}/[P(\bar{B})]^{1/2}$.

COROLLARY 4.1. *If Y and Z are the indicator random variables of events A and B respectively, then*

$$E(Y|Z) = P(AB)Z/P(B) + P(A\bar{B})\bar{Z}/P(\bar{B}).$$

We note that $E(Y|Z=1)=P(AB)/P(B)$, the usual formula for the conditional probability of A given B.

The concept of independence of random variables may be introduced as follows. Suppose X and Z are random variables, such that the orthogonal projection on $\mathcal{V}_Z$ of every function $f(X)$ is a constant random variable: $E(f(X)|Z) =E(f(X))U$. Then $f(X)-E(f(X))U$ is orthogonal to $g(Z)$ for every function g, so that

$$\sum_{i=1}^{n} [f(x_i) - E(f(X))]g(z_i)m_i = 0,$$

which implies that

$$\sum_{i=1}^{n} [f(x_i) - E(f(X))][g(z_i) - E(g(Z))]m_i = 0$$

for every pair of functions f, g. This in turn implies

$$\sum_{i=1}^{n} f(x_i)[g(z_i) - E(g(Z))]m_i = 0;$$

hence $E(g(Z)|X)=E(g(Z))U$. That is, if $E(f(X)|Z)=E(f(X))U$ for every function f, then also $E(g(Z)|X)=E(g(Z))U$ for every function g.

DEFINITION 4.2. The random variables X and Z are *independent* if $E(f(X)|Z) =E(f(X))U$ for every function f (or equivalently, $E(g(Z)|X)=E(g(Z))U$ for every function g). We note that the functions of U are just the constant random variables, so that U is independent of every random variable, even of itself.

In Example 2.2, let Y denote the indicator of the event "point up on first toss" and Z the indicator of the event "point up on second toss": $Y=(1, 1, 0, 0)$, $Z=(1, 0, 1, 0)$. One readily verifies that these two random variables are independent, as indeed one's sense of the fitness of things demands they should be.

THEOREM 4.7. $E(E(Y|X)) = E(Y)$.

Proof. $E(Y|X)$ is just the orthogonal projection of Y on $\mathcal{V}_X$, and $E(E(Y|X))U$ is just the orthogonal projection of $E(Y|X)$ onto a subspace of $\mathcal{V}_X$, namely the subspace of all constant random variables. But $E(Y)U$ is just the orthogonal projection of Y on the subspace of all constant random variables and, by Theorem 4.5, $E(E(Y|X)) = E(Y)$.

THEOREM 4.8. $E(XY|X) = XE(Y|X)$. This is immediate from Theorem 4.4.

COROLLARY 4.2. *If X and Y are independent then* $E(XY) = E(X)E(Y)$.

Using the formula $V(Z) = E(Z^2) - [E(Z)]^2$ one also obtains the following corollary.

COROLLARY 4.3. *If X and Y are independent, then* $V(X+Y) = V(X) + V(Y)$.

Events A and B are called *independent* if their indicator random variables are independent.

COROLLARY 4.4. *Events A and B are independent if and only if* $P(AB) = P(A)P(B)$.

Proof. Let Y and Z be indicator random variables of A and B respectively. Then YZ is the indicator of AB, and if Y and Z are independent we have $P(AB) = E(YZ) = E(Y)E(Z) = P(A)P(B)$. Conversely, if $P(AB) = P(A)P(B)$, one verifies also that $P(A\bar{B}) = P(A)P(\bar{B})$, so that Corollary 4.1 gives $E(Y|Z) = P(A)Z + P(A)\bar{Z} = P(A)U$. If f is a function, set $a = f(1) - f(0)$, $b = f(0)$, so that $f(Y) = aY + bU$ and $E(f(Y)) = aP(A) + b$. Then, using Theorem 4.3, $E(f(Y)|Z) = aE(Y|Z) + bE(U|Z) = aP(A)U + bU = E(f(Y))U$, so that Y and Z are independent.

Finally, we give in Theorems 4.9 and 4.10 the standard noncalculus solution of the least square linear regression problem.

THEOREM 4.9. *If X and Y are given random variables then $\|Y - aX\|^2$ is minimized for real a by* $a = (X, Y)/\|X\|^2$.

Proof.

$$\begin{aligned} \|Y - aX\|^2 &= (Y - aX, Y - aX) \\ &= (Y, Y) - 2a(X, Y) + a^2(X, X) \\ &= (Y, Y) - (X, Y)^2/(X, X) + (X, X)\left(a - \frac{(X, Y)}{(X, X)}\right)^2 \end{aligned}$$

which implies, as before, that the minimum is achieved for $a = (X, Y)/(X, X)$.

Theorem 3.1 is readily seen to be a special case of this theorem in which one replaces X by U.

THEOREM 4.10. *If X and Y are given random variables, then $\|Y - aX - bU\|^2$ is minimized for real a and b by* $a = (X - (X, U)U, Y - (Y, U)U)/\|X - (X, U)U\|^2$, *and* $b = (Y, U) + a(X, U)$.

Proof. Set $f(a, b)=\|Y-aX-bU\|^2$. By Theorem 3.1, for fixed a, $f(a, b)$ achieves its minimum for $b=[(Y, U)-a(X, U)]$. The minimum value is $\|[Y-(Y, U)U]-a[X-(X, U)U]\|^2$, and by Theorem 4.9, this function of a achieves its minimum for

$$a = (X - (X, U)U,\ Y - (Y, U)U)/\|X - (X, U)U\|^2.$$

The work of the second author was done as a participant in a program of undergraduate independent study sponsored by the National Science Foundation (Grant NSF 21659).

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THE KURATOWSKI CLOSURE AXIOMS

SHAIR AHMAD, South Dakota State University

In a note [1], Harary introduced the concepts of "very" independent and "absolutely" independent axioms by using a "never holding" type of denial of an axiom. It is the purpose of this paper to show that the Kuratowski closure axioms can be made not only "very" independent, but "absolutely" independent.

If A and B are subsets of a space X and c is a mapping of X into itself, then the Kuratowski closure axioms are:

I. $\emptyset^c=\emptyset$ III. $A^{cc}=A^c$

II. $A\subset A^c$ IV. $(A\cup B)^c=A^c\cup B^c$.

In this case c is called the closure operator.

An equivalent set of axioms is:

(i) $\emptyset^c=\emptyset$, (ii) $A\cup A^{cc}\subset A^c$, (iii) $(A\cup B)^c=A^c\cup B^c$.

It is easily seen that axioms II and III imply (ii), and (ii) implies II. To show the equivalence, we will show that (ii) implies III. From (ii), we have $A^{cc}\subset A^c$. Since $A^c\cup A^{ccc}\subset A^{cc}$, we also have $A^c\subset A^{cc}$. Therefore, $A^{cc}=A^c$, and the equivalence is established.

In order to have an "absolutely" independent set of axioms, one must delete

Proof. Set $f(a, b) = \|Y - aX - bU\|^2$. By Theorem 3.1, for fixed a, $f(a, b)$ achieves its minimum for $b = [(Y, U) - a(X, U)]$. The minimum value is $\|[Y-(Y, U)U] - a[X-(X, U)U]\|^2$, and by Theorem 4.9, this function of a achieves its minimum for

$$a = (X - (X, U)U, Y - (Y, U)U)/\|X - (X, U)U\|^2.$$

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In a note [1], Harary introduced the concepts of "very" independent and "absolutely" independent axioms by using a "never holding" type of denial of an axiom. It is the purpose of this paper to show that the Kuratowski closure axioms can be made not only "very" independent, but "absolutely" independent.

If A and B are subsets of a space X and c is a mapping of X into itself, then the Kuratowski closure axioms are:

I. $\emptyset^c = \emptyset$ III. $A^{cc} = A^c$

II. $A \subset A^c$ IV. $(A \cup B)^c = A^c \cup B^c$.

In this case c is called the closure operator.

An equivalent set of axioms is:

(i) $\emptyset^c = \emptyset$, (ii) $A \cup A^{cc} \subset A^c$, (iii) $(A \cup B)^c = A^c \cup B^c$.

It is easily seen that axioms II and III imply (ii), and (ii) implies II. To show the equivalence, we will show that (ii) implies III. From (ii), we have $A^{cc} \subset A^c$. Since $A^c \cup A^{ccc} \subset A^{cc}$, we also have $A^c \subset A^{cc}$. Therefore, $A^{cc} = A^c$, and the equivalence is established.

In order to have an "absolutely" independent set of axioms, one must delete from each axiom all cases that can be proven true, independent of that axiom. With this in mind, we state axioms (i), (ii), and (iii) as follows:

(1) $\emptyset^c=\emptyset$

(2) $A\cup A^{cc}\subset A^c$ for all $A\neq\emptyset$

(3) $(A\cup B)^c=A^c\cup B^c$ if $A\not\subset B$, $B\not\subset A$ and $A\cup B\neq X$.

Axioms (1), (2), and (3) certainly follow from (i), (ii), and (iii). We note that (i) follows from (1), and (ii) follows from (1) and (2). To establish the fact that (1), (2), and (3) are equivalent to (i), (ii), and (iii), we show that (iii) follows from (1), (2) and (3).

a. *Suppose* $A\subset B$ *and* $B\neq X$. If $A=\emptyset$, (iii) follows from (1). If $A=B$, (iii) follows directly from set theory. If $A\neq\emptyset$, and $A\neq B$, then let $D=B\cap A'$ so that $A\cup D=B$, $A\not\subset D$ and $D\not\subset A$. From (3), we have $B^c=(A\cup D)^c=A^c\cup D^c$. Therefore, $A^c\subset B^c$, or $A^c\cup B^c\subset B^c=(A\cup B)^c$. On the other hand, $(A\cup B)^c=B^c$ implies that $(A\cup B)^c\subset A^c\cup B^c$. Thus, we have $(A\cup B)^c=A^c\cup B^c$, and (iii) follows. The case $B\subset A$ is similar.

b. *Suppose* $(A\cup B)=X$. It follows from (2) that $X^c=(A\cup B)^c=X$. Also, since $A\cup A^{cc}\subset A^c$ and $B\cup B^{cc}\subset B^c$, it follows that $A\cup B=X\subset A^c\cup B^c$. Thus, $(A\cup B)^c=X=A^c\cup B^c$, and (iii) holds.

If we note that the "never holding" denials of (1), (2) and (3) in the sense used by Harary are

$(\bar{1})$ $\emptyset^c\neq\emptyset$

$(\bar{2})$ There is at least one nonempty set A in X, and $A\cup A^{cc}\not\subset A^c$ whenever $A\neq\emptyset$.

$(\bar{3})$ There are sets A and B in X such that $A\not\subset B$, $B\not\subset A$ and $A\cup B\neq X$; further $(A\cup B)^c\neq A^c\cup B^c$ whenever $A\not\subset B$, $B\not\subset A$ and $A\cup B\neq X$.

We see that the following examples show that (1), (2) and (3) are "absolutely" independent with respect to the set $X=\{x, y, z\}$.

Mapping	*Axioms*		
$A^c=A$ for all A	(1)	(2)	(3)
$A^c=X$ for all A	$(\bar{1})$	(2)	(3)
$A^c=\emptyset$ for all A	(1)	$(\bar{2})$	(3)
$\{x\}^c=\{x\}$, $\{y\}^c=\{y\}$, $\{z\}^c=\{z\}$, $\emptyset^c=\emptyset$ $A^c=X$ for all other A	(1)	(2)	$(\bar{3})$
$\emptyset^c=X$, $A^c=\emptyset$ for all $A\neq\emptyset$	$(\bar{1})$	$(\bar{2})$	(3)
$\emptyset^c=\emptyset$, $A^c=A'$ for all $A\neq\emptyset$	(1)	$(\bar{2})$	$(\bar{3})$
$\{x\}^c=\{x\}$, $\{y\}^c=\{y\}$, $\{z\}^c=\{z\}$, $\emptyset^c=X$ $A^c=X$ for all other A	$(\bar{1})$	(2)	$(\bar{3})$
$A^c=A'$ for all A	$(\bar{1})$	$(\bar{2})$	$(\bar{3})$

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COMBINATORIAL HOMOTOPY THEORY AND A NEW PROOF THAT THE SECOND HOMOTOPY GROUP OF THE CIRCLE IS TRIVIAL

HARRY GONSHOR, Rutgers-The State University

I. We give here a new combinatorial definition for the second homotopy group of a simplicial complex. Though the definition generalizes easily to higher homotopy groups, it is convenient for the sake of notational simplicity to limit ourselves to the second homotopy group. The approach is less general and less sophisticated than that of D. M. Kan [1]. Using this definition we shall prove that the second homotopy group of the circle vanishes. We feel that the proof is interesting because of its extremely elementary nature using no topology or algebra! (The only place where topology would be required is in the proof of the equivalence of our definition of the homotopy groups and the usual one [2]. This is essentially a consequence of the simplicial approximation theorem.)

II. We consider an abstract simplicial complex, i.e., a set S of elements called vertices and a distinguished collection K of subsets called simplices with the following properties:

a. $A \in k \Rightarrow A$ is finite

b. $p \in S \Rightarrow (p) \in K$

c. $A \in K$ and $B \subset A \Rightarrow B \in K$.

We now single out a vertex which we call e. We define a homotopy chain as a finite double sequence $f(i, j)$ of vertices where $0 \leqq i \leqq m$, $0 \leqq j \leqq n$, satisfying $f(0, j) = f(m, j) = f(i, 0) = f(i, n) = e$ for all i and j, and the set $[f(i, j), f(i+1, j), f(i, j+1), f(i+1, j+1)]$ is a simplex for all i and j, where the f's are meaningful. (In the special case where $i = m$ or $j = n$ simply discard the meaningless f's.) We shall use the notation $m(f)$ and $n(f)$ to indicate the dependence of m and n on the homotopy chain f.

A homotopy chain g is a row extension of a homotopy chain f if $m(g) = m(f) + 1$, $n(g) = n(f)$ and furthermore there exists an i_0 such that $0 \leqq i_0 \leqq m(f)$ and such that $g(i, j) = f(i, j)$ for $i \leqq i_0$ and $g(i, j) = f(i-1, j)$ for $i > i_0$. Intuitively we can think of g as being obtained from f by repeating a "row." Similarly we define a column extension by reversing the role of i and j. Furthermore we define

$$g \sim f \Leftrightarrow m(g) = m(f), \qquad n(g) = n(f),$$

f and g differ only for one pair (i, j), and furthermore for all i and j the set

$$[f(i, j), f(i+1, j), f(i, j+1), f(i+1, j+1), g(i, j), g(i+1, j), g(i, j+1), g(i+1, j+1)]$$

is a simplex. Finally we define $g \cong f \Leftrightarrow$ there exists a sequence $f_0, f_1, \cdots, f_n$ such that $f_0 = f$, $f_n = g$, and for every two consecutive f's, f_i and f_{i+1}, either $f_i \sim f_{i+1}$ or one of these is a row or column extension of the other. Clearly $\cong$ is an equivalence relation.

The equivalence classes of homotopy chains may be made into a group as follows.

If f and g are representatives of two classes, then as a representative of the sum we take h where $m(h)=\max[m(f), m(g)]$, $n(h)=n(f)+n(g)$, $h(i, j)=f(i, j)$ if $i \leqq m(f)$ and $j \leqq n(f)$, $h(i, j)=g[i, j-n(f)]$ if $i \leqq m(g), j>n(f)$, and $h(i, j)=e$ otherwise.

Intuitively we can think of h as obtained by placing g to the right of f and filling up the missing positions with e if necessary to obtain a rectangular array. Incidentally, although we use the language of functions for the sake of precision, greater insight is obtained by thinking in terms of arrays. This is analogous to the situation in calculus where it is convenient to express a sequence in the form $a_1, a_2, a_3, \cdots$ instead of using functional notation.

In order to show that addition is well defined it suffices to show that

$$f_1 \cong f_2 \Rightarrow f_1 + g \cong f_2 + g.$$

A similar method would apply to the case where the second addend is altered and the proof is completed by transitivity. It suffices to consider the three cases $f_1 \sim f_2$, one is a column extension of the other, and one is a row extension of the other. The first two cases are indeed trivial. The last case is tricky because if f_2 is a row extension of f_1, then f_2+g is not necessarily a row extension of f_1+g. The idea is to make use of $\sim$ as follows: If f_2 has the ith row of f_1 repeated we can go to f_3+g where f_3 has the $(i+1)$-st row of f_1 repeated. By induction we get to f_4+g where f_4 has the last row of f_1 repeated. If $m(f_1) < m(g)$, this is exactly f_1+g. Otherwise f_4+g is a row extension of f_1+g.

We now check that the equivalence classes form a group under addition. The associative law is trivial to verify. Furthermore, any homotopy chain g satisfying $g(i, j)=e$ for all i and j is a representative of the identity. If f is a representative of a class, then as a representative of the inverse we can take g where $m(g)=m(f)$, $n(g)=n(f)$ and $g(i, j)=f(i, n(g)-j)$.

Intuitively g is obtained from f by reversing the order of the columns. The key trick required in justifying this is the fact that if two columns which are alike are separated by a single column, then that column can be changed to the common columns by using $\sim$.

Corresponding to a homotopy chain, one can obtain an element of the second homotopy group in a natural way. It can be shown that this correspondence is an isomorphism.

III. Several theorems about homotopy groups, such as their commutativity, may be proved using this definition. In most cases the proofs are analogous to the usual ones. One case for which the proof is quite different is that of the theorem that the second homotopy group of the circle is trivial.

In order to prove this, it suffices to show that any homotopy chain is equivalent to a chain all of whose values are e. We shall show, in fact, that an equivalence can be set up using $\sim$ only, i.e., row or column extensions are not used.

A simplicial complex corresponding to the circle is the set of vertices e, a, and b with the class of simplices being the class of all subsets other than (e, a, b). We can now rephrase the problem so that it resembles a recreational puzzle.

We consider a rectangular array of points (i, j) where i and j are integers satisfying $0 \leqq i \leqq m$ and $0 \leqq j \leqq n$. Suppose that to each point (i, j) is assigned a vertex $f(i, j)$ from the set (e, a, b) such that every "little square" (a subset of the form $[(i, j), (i+1, j), (i, j+1), (i+1, j+1)]$) has at most two distinct vertices assigned to its members, and e is assigned to every point of the boundary (all vertices (i, j) where $i=0$, $i=m$, $j=0$, or $j=n$). Such an assignment will be called a "position."

We want to show that all the assigned vertices can be changed to e by a succession of "legal moves," where by a legal move we mean a change of a vertex at a single point to a new position such that for every little square containing the point, the set consisting of the vertices at the three other points of the little square, the old vertex at the point, and the new vertex at the point consists of at most two distinct vertices. For example, this can trivially be done if the position contains at most two distinct vertices, since the change of each vertex to e one at a time is clearly a legal move.

We show this by contradiction. Suppose that there exists a position for which not all assigned vertices can be changed to e by legal moves. We order the points (i, j) lexicographically so that (i_1, j_1) precedes (i_2, j_2) if $i_1<i_2$ or $i_1=i_2$ and $j_1<j_2$ and consider the sequence of vertices obtained. Then there is a point (i, j) at which a third new vertex is obtained. (Such a point exists by the preceding remark.) Now for all positions which can be obtained from the given one by legal moves choose one which maximizes the point (i, j). Then it follows that it is impossible by a sequence of legal moves to obtain a position with the vertex at (i, j) changed and the vertices at all smaller points left unaltered. Let $f(i, j)=x$. Note that x is either a or b and that $i, j>0$. Since x is a new vertex $f(i-1, j)\neq x$. Let $f(i-1, j)=y$. Again because x is a new vertex, and because of the property satisfied by little squares, $f(i-1, j-1)=f(i, j-1)=y$. Now by considering the little square $[(i-1, j), (i, j), (i-1, j+1), (i, j+1)]$ it follows that $f(i-1, j+1)$ and $f(i, j+1)$ are either x or y. Similarly $f(i+1, j-1)$ and $f(i+1, j)$ are either x or y. Now consider $f(i+1, j+1)$. If $f(i+1, j+1)$ were either x or y then $f(i, j)$ could be changed to y by a legal move contrary to hypothesis. Thus $f(i+1, j+1)=z$ where $z\neq x$ and $z\neq y$. By considering the little square

$$[(i, j), (i+1, j), (i, j+1), (i+1, j+1)]$$

it follows that $f(i+1, j)=f(i, j+1)=x$. Also it is impossible by legal moves to change the vertex at $(i+1, j+1)$ to x keeping the vertices at all smaller points fixed, for then the vertex at (i, j) could be changed in the following move. Thus we have a situation at $(i+1, j+1)$ which is similar to the one at (i, j). By similar reasoning we obtain $f(i+1, j+1)=f(i+2, j+1)=f(i+1, j+2)\neq f(i+2, j+2)$. Thus by induction $f(i+k, j+k+1)\neq f(i+k+1, j+k+1)$ for all $k\geqq 0$. However, this is false when $j+k+1=n$ since both values are e. (The latter is valid only if $n-j\leqq m-i$. Otherwise the role of i and j must be reversed.)

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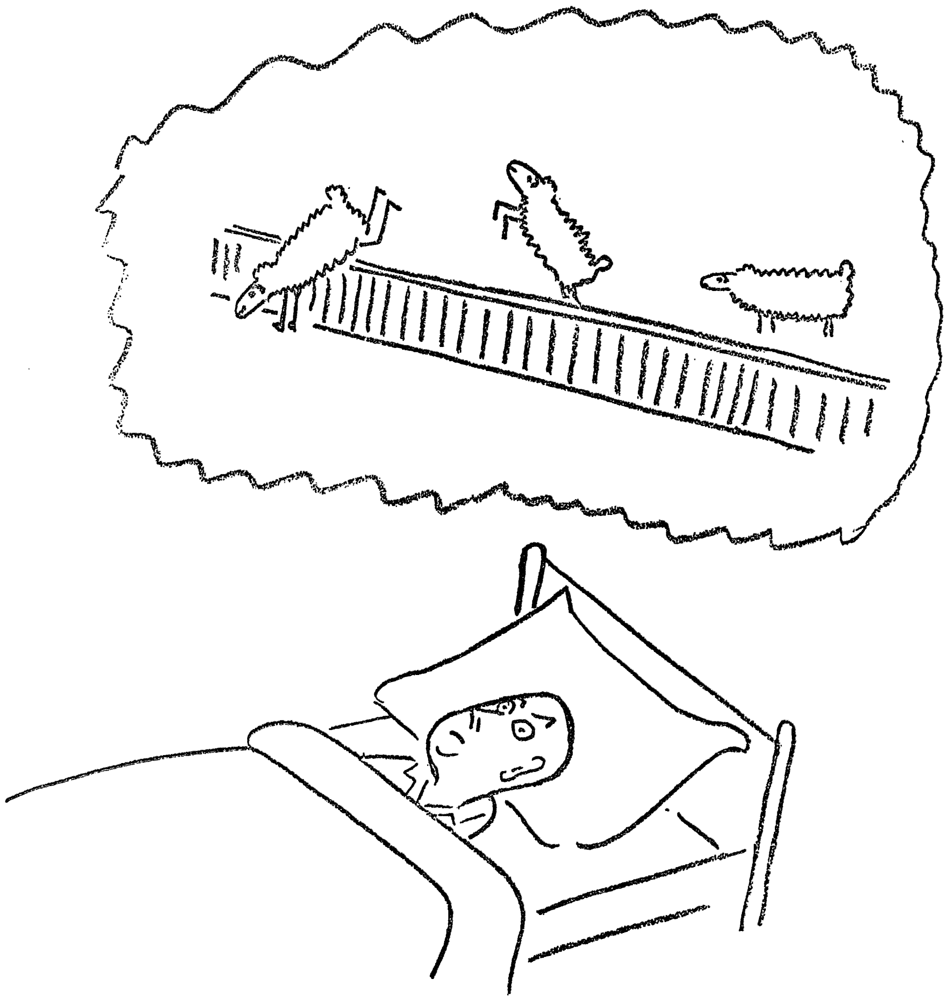

"$\ldots, \omega+2, \omega+3, \omega+4$"

R. C. BUCK

FACTORIZATION OF LINEAR SECOND ORDER DIFFERENTIAL OPERATORS

J. H. HEINBOCKEL, North Carolina State University

For the second order linear differential equation with variable coefficients

$$(1) \qquad [P_0(x)D^2 + P_1(x)D + P_2(x)]y = 0, \qquad \left(D = \frac{d}{dx}\right)$$

let us assume that P_0, P_1, and P_2 are continuous functions in some interval and that the derivatives P_0'' and P_1' exist in this interval. In the following, we will study the factorization of (1) into the form

$$(2) \qquad [P_0(x)D + M(x)][D + N(x)]y = 0.$$

Explicit expressions for $M(x)$ and $N(x)$ are obtained under the assumption that either a nonzero solution of (1) is known or a nonzero solution of the adjoint differential equation associated with (1) is known.

If (1) is factorable into the form (2) then it is readily verified that M and N satisfy the equations

$$(3) \qquad P_0N + M = P_1, \qquad P_0N' + MN = P_2.$$

Eliminating M from the equations (3) produces the generalized Riccati equation

$$(4) \qquad P_0N' = P_0N^2 - P_1N + P_2.$$

It is well known that the solutions of (1) and (4) are related by the substitution $N = -y'/y$, (cf. [1]). Thus, if $y_1 = y_1(x)$ is a known solution of (1), then $N_1 = -y_1'/y_1$ is a solution of (4) and the general solution of (4) may be obtained by two quadratures as follows:

Substituting $N = 1/u - y_1'/y_1$ into (4), we find that u satisfies the linear first order differential equation

$$(5) \qquad u' - \left(\frac{P_1}{P_0} + \frac{2y_1'}{y_1}\right)u = -1.$$

Let $\Phi(x) = \int P_1(x)dx/P_0(x)$; then the general solution of (5) may be expressed as

$$(6) \qquad u = u_0(x) + Ku_1(x),$$

where $u_0(x) = -u_1(x)\int dx/u_1(x)$, $u_1(x) = e^{\Phi}y_1^2$ and where K is an arbitrary integration constant. The general solution of (4) has the form

$$(7) \qquad N(x) = \frac{y_1 - y_1'(u_0 + Ku_1)}{y_1(u_0 + Ku_1)}$$

and consequently from (3) we have also $M(x) = P_1 - P_0N$. These are the general solutions of (3).

Alternatively, if one knows two linearly independent solutions of (1), say Y_1 and Y_2, then the general solution of (1) is any linear combination of Y_1 and Y_2 and the corresponding general solution of (4) may be expressed as

$$N(x) = \frac{-y'}{y} = \frac{-c_1 Y_1' - c_2 Y_2'}{c_1 Y_1 + c_2 Y_2}, \tag{8}$$

where c_1 and c_2 are arbitrary constants. We may correlate the two solutions (7) and (8) by letting $c_1 = K$ and appropriately choosing the constant c_2. We then have that

$$\frac{-KY_1' - c_2 Y_2'}{KY_1 + c_2 Y_2} = \frac{y_1 - y_1'(u_0 + Ku_1)}{y_1(u_0 + Ku_1)}, \tag{9}$$

for all constants K. In particular as $K \to \infty$, we obtain

$$\frac{-Y_1'}{Y_1} = \frac{-y_1'}{y_1}, \tag{10}$$

which implies $Y_1 = y_1$; (i.e., the above correlation between (7) and (8) was chosen to yield this result). Further, as $K \to 0$ in (9), there results

$$\frac{-Y_2'}{Y_2} = \frac{1}{u_0} - \frac{Y_1'}{Y_2},$$

which implies

$$\frac{d}{dx}\left(\frac{Y_2}{Y_1}\right) = \frac{-1}{u_0}\left(\frac{Y_2}{Y_1}\right). \tag{11}$$

An integration of equation (11) produces Abel's identity

$$Y_2 = AY_1 \int \frac{e^{-\Phi} dx}{Y_1^2},$$

where A is an arbitrary constant. The above results are well known and may be derived using other types of arguments (cf. [1], [2]).

Let $z_1 = z_1(x)$ be a nonzero solution of the adjoint differential equation associated with (1)

$$P_0 z'' + (2P_0' - P_1)z' + (P_0'' - P_1' + P_2)z = 0; \tag{12}$$

then

$$N_0 = \frac{P_1 - P_0'}{P_0} - \frac{z_1'}{z_1}$$

is a particular solution of (4), as is verified by substituting N_0 into (4). It appears that this fact has not been considered previously. Since N_0 is a particular solution of (4), the general solution may be obtained by using the substitution $N = 1/V + N_0$ in (4). This substitution produces the equation

$$V' + \left(\frac{P_1}{P_0} - \frac{2P_0'}{P_0} - \frac{2z_1'}{z_1}\right)V = -1. \tag{13}$$

Letting $\Phi=\int P_1(x)/P_0(x)dx$, we may express the general solution of (13) as $V=V_0(x)+KV_1(x)$ where

$$V_0(x) = -V_1(x)\int \frac{dx}{V_1(x)}, \qquad V_1(x) = e^{-\Phi}P_0^2 z_1^2$$

and K is an arbitrary constant of integration. Hence the general solution to (4) may be expressed as

$$N(x) = \frac{1+V_0N_0+KV_1N_0}{V_0+KV_1} \tag{14}$$

and, from (3), $M(x)=P_1-P_0N$. These are the general solutions of (3) in terms of a known solution of the adjoint differential equation associated with (1).

The relation between the solution y of (1) and z of (12) is given by

$$\frac{-y'}{y} = N = \frac{P_1-P_0'}{P_0} - \frac{z'}{z}. \tag{15}$$

An integration of this equation produces the fact that

$$y = P_0ze^{-\Phi}. \tag{16}$$

Observe that if (1) is self adjoint (i.e., $P_1=P_0'$) then (16) is an identity. It is interesting to note that if $L(D)$ denotes the operator $L(D)=P_0D^2+P_1D+P_2$ and $L^*(D)$ denotes the adjoint operator $L^*(D)=P_0D^2+(2P_0'-P_1)D+(P_0''-P_1'+P_2)$, then Lagrange's formula

$$yL^*(D)z - zL(D)y = \frac{d}{dx}\left[P_0zy\left\{\frac{z'}{z}-\frac{y'}{y}-\frac{(P_1-P_0')}{P_0}\right\}\right]$$

implies the relation (15).

Let us finally remark that if $Dy+N(x)y\equiv 0$, then

$$y = A\exp\left\{\int \frac{1+V_0N_0+KV_1N_0}{V_0+KV_1}dx\right\},$$

where A and K are arbitrary constants, is a solution of (1). If $Dy+N(x)y=W(x)\neq 0$, then the general solution of (1) may be obtained by solving the linear first order equations

$$P_0DW+M(x)W=0, \qquad Dy+N(x)y=W.$$

Thus, if one solution of the adjoint equation is known, an alternate method of obtaining the solution to (1) is the method of factorization of operators.

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HOW TO SOLVE CYCLIC DIFFERENTIAL EQUATIONS

EDGAR KARST, University of Oklahoma

Cyclic (symmetric) differential equations are a typical textbook favorite. Their solution by conventional means is tedious to the student, but enjoyable to the author of the book because often it is he who has constructed the problem from a beautifully simple answer by differentiation. It is the purpose of this paper to present detailed solutions of one example:

(a) from the student's point of view,

(b) from the standpoint of a master mind (Goursat),

(c) from the viewpoint of a numerical analyst who plays with cyclic patterns.

The example shall be to solve

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0 \tag{1}$$

as suggested as a problem in [2, p. 33] presuming the knowledge of the integrability conditions, $X\cdot\text{curl}\, X = 0$, and Natani's method in case (a), Bertrand's method in case (b), and a workable rule in case (c).

(a) To verify the integrability of (1) we note that $X = (P, Q, R) = (y^2+yz+z^2, z^2+zx+x^2, x^2+xy+y^2)$ so that

$$\text{curl}\, X = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 + zx + x^2 & x^2 + xy + y^2 \end{vmatrix} i$$

$$- \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ y^2 + yz + z^2 & x^2 + xy + y^2 \end{vmatrix} j + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ y^2 + yz + z^2 & z^2 + zx + x^2 \end{vmatrix} k$$

$$= (x + 2y - 2z - x)i - (2x + y - y - 2z)j + (z + 2x - 2y - z)k$$

$$= 2(y - z)i - 2(x - z)j + 2(x - y)k = 2(y - z, -x + z, x - y).$$

But

$$X\cdot\text{curl}\, X = 2[P(y - z) + Q(-x + z) + R(x - y)]$$

$$= 2(y^3 + y^2z + yz^2 - y^2z - yz^2 - z^3 - xz^2 - x^2z - x^3 + z^3 + xz^2 + x^2z + x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3) = 0.$$

Hence the differential equation (1) is integrable.

Let $x = uz$ and $y = vz$; then $dx = udz + zdu$ and $dy = vdz + zdv$. Therefore, $z^2(v^2 + v + 1)(udz + zdu) + z^2(1 + u + u^2)(vdz + zdv) + z^2(u^2 + uv + v^2)dz = 0$ or $z(v^2 + v + 1)du + z(1 + u + u^2)dv + (u + v + 1)(u + uv + v)dz = 0$. Dividing by $z(v^2+v+1)(1+u+u^2)$ we obtain

$$\text{(2)} \qquad \frac{du}{1+u+u^2}+\frac{dv}{v^2+v+1}+\frac{(u+v+1)(u+uv+v)}{z(v^2+v+1)(1+u+u^2)}dz = 0.$$

To find its primitive (solution), let us use Natani's method. Let $dz=0$. Then equation (2) is reduced to

$$\frac{du}{1+u+u^2}+\frac{dv}{v^2+v+1}=0 \text{ or } \frac{du}{(\sqrt{3}/2)^2+(1/2+u)^2}+\frac{dv}{(v+1/2)^2+(\sqrt{3}/2)^2}=0.$$

Now let $u+1/2=s$ and $v+1/2=t$; then $du=ds$ and $dv=dt$. Hence,

$$\frac{ds}{(\sqrt{3}/2)^2+s^2}+\frac{dt}{t^2+(\sqrt{3}/2)^2}=0$$

which has the solution

$$\text{(3)} \qquad (2/\sqrt{3})\tan^{-1}(2u+1)/\sqrt{3}+(2/\sqrt{3})\tan^{-1}(2v+1)/\sqrt{3}=f(z).$$

Substituting $v=0$ in (2) we obtain

$$\frac{du}{1+u+u^2}+\frac{(u+1)u}{z(1+u+u^2)}dz=0$$

and multiplying by $(1+u+u^2)/u(u+1)$ we get

$$\frac{du}{u(u+1)}+\frac{dz}{z}=0 \quad \text{or} \quad \frac{du}{u}-\frac{du}{u+1}+\frac{dz}{z}=0$$

which has the solution $\ln u-\ln(u+1)+\ln z=\ln c$ or $uz/(u+1)=c$. Since we have to use $2u+1$ later we transform this to $2u+1=(z+c)/(z-c)$. This solution must be the form assumed when $v=0$ is substituted in (3) yielding

$$(2/\sqrt{3})\tan^{-1}(2u+1)/\sqrt{3}+(2/\sqrt{3})\tan^{-1}(1/\sqrt{3})=f(z).$$

Hence, substituting $2u+1=(z+c)/(z-c)$, we get

$$(2/\sqrt{3})\tan^{-1}(z+c)/(z-c)\sqrt{3}+(2/\sqrt{3})\tan^{-1}(1/\sqrt{3})=f(z).$$

Substituting this expression into (3) we obtain, having multiplied by $\sqrt{3}/2$,

$$\tan^{-1}(2u+1)/\sqrt{3}+\tan^{-1}(2v+1)/\sqrt{3}=\tan^{-1}(z+c)/(z-c)\sqrt{3}+\tan^{-1}(1/\sqrt{3})$$

and making use of the addition formula

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}(x+y)/(1-xy)$$

we obtain

$$\frac{\sqrt{3}(2u+1+2v+1)}{3-(2u+1)(2v+1)}=\frac{\sqrt{3}(z+c+z-c)}{3z-3c-z-c}$$

which reduces in terms of x, y and z to the answer $yz+zx+xy=c(x+y+z)$.

(b) From [1, p. 233]: The condition $P(Q_z-R_y)+Q(R_x-P_z)+R(P_y-Q_x)=0$ is satisfied, and the linear equation

$$X(f) = (Q_z - R_y)f_x + (R_x - P_z)f_y + (P_y - Q_x)f_z = 0$$

is here

$$(z - y)f_x + (x - z)f_y + (y - x)f_z = 0.$$

The corresponding system of differential equations,

$$dx/(z - y) = dy/(x - z) = dz/(y - x),$$

gives easily the two integrable combinations

$$d(x + y + z) = 0, \qquad xdx + ydy + zdz = 0.$$

Hence we may take $u=x+y+z$, $v=x^2+y^2+z^2$, and the values of the factors λ and μ derived from the equations

$$P = \lambda u_x + \mu v_x, \qquad Q = \lambda u_y + \mu v_y, \qquad R = \lambda u_z + \mu v_z,$$

are

$$\lambda = x^2 + y^2 + z^2 + xy + yz + zx = (u^2 + v)/2,$$
$$\mu = -(x + y + z)/2 = -u/2.$$

The transformed equation in u and v is, therefore,

$$(u^2 + v)du - udv = 0, \quad \text{or} \quad du = (udv - vdu)/u^2 = d(v/u).$$

It follows that the general integral is $u-v/u=c$, or, returning to the variables x, y, z, we obtain, as before $xy+yz+zx=c(x+y+z)$.

(c) Let us establish what we may call neither a theorem nor a conjecture, but a workable rule:

Cyclic differential equations have cyclic patterns. From this it follows at once that by differentiating a cyclic pattern in x, y, z on the left side and a constant, c, at the right side of an equation, we should obtain a cyclic differential equation.

A pattern of 3 variables has only a few variations. Hence, let us start with the simplest ones:

(A) Differentiation of $x+y+z=c$ yields $dx+dy+dz=0$.

(B) Differentiation of $xyz=c$ yields $xydz+yzdx+xzdy=0$.

(C) Differentiation of $xy+yz+zx=c$ yields $(x+y)dz+(y+z)dx+(z+x)dy=0$.

(D) Differentiation of $xyz/(x+y+z)=c$ yields $xy(x+y)dz+yz(y+z)dx+zx(z+x)dy=0$.

(E) Differentiation of $(xy+yz+zx)/(x+y+z)=c$ yields

$$(x^2 + xy + y^2)dz + (y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy = 0. \tag{1}$$

Hence, in a few minutes we have found the solution of (1) by just playing with cyclic patterns and a workable rule. This rule helps also to detect incomplete solutions. For example, one could have found and checked by differentiation that $(x^2-y^2)/(z^2-y^2)=c$ is "the" solution of $x(y^2-z^2)dx+y(z^2-x^2)dy+z(x^2-y^2)dz=0$. But $(x^2-y^2)/(z^2-y^2)$ is not a cyclic pattern. Hence, the complete solution, being a cyclic pattern is

$$(x^2 - y^2)/(z^2 - y^2) = (y^2 - z^2)/(x^2 - z^2) = (z^2 - x^2)/(y^2 - x^2) = c.$$

The author would like to suggest that solutions of cyclic differential equations should be listed like regular integrals. This would make the student happier, the textbook more economic, and the authors of those texts more and more bewildered.

References

1. E. Goursat, E. R. Hedrick, and O. Dunkel, Differential equations, Ginn, Boston, 1917.
2. I. N. Sneddon, Elements of partial differential equations, McGraw-Hill, New York, 1957.

REPRESENTING NUMBERS USING ONLY ONE 4

DONALD E. KNUTH, California Institute of Technology

In the Mathematical Games section of *Scientific American*, January, 1964, Martin Gardner asks his readers to try their hand at the well-known problem of representing the number 64 using precisely four, three, and two 4's, respectively, combined with arithmetic operations. The "two-4" problem was first posed by J. A. Tierney in 1944 (see [1]), and some of the answers given were

$$64 = \sqrt{\sqrt{\sqrt{(4^{4!})}}} = \sqrt{\ (4^{\Gamma(4)})} = 4^{[\ln 4!]} = \phi(\phi(4^4)) = \phi(4!)^{\sqrt{4}} = 4^{\sqrt{(\pi(4!))}}.$$

It is the purpose of this note to show that the number 64 can be represented with only *one* 4, and indeed this may be true of all positive integers.

Using only the three operations $\surd$, $[\ ]$, and $!$, and the single digit 4, we have

$$\begin{aligned}64 = {} & [\surd\surd\surd\surd\surd\surd\surd\surd\surd[\surd\surd\surd\surd\surd\surd\surd\surd\surd[\surd\surd\surd\surd\surd\surd\surd\surd\surd\surd\surd\surd\surd \\ & [\surd\surd\surd\surd\surd\surd\surd\surd[\surd\surd\surd\surd\surd\surd\surd\surd\surd\surd\surd \\ & [\surd[\surd[\surd\surd\surd\surd\surd(4!)!]!]!]!]!]!]!]!].\end{aligned}$$

(In this representation, the notation $\surd x!$ stands for $\surd(x!)$, not $(\surd x)!$.)

To prove this we will compute successively the quantities inside each of the nested brackets. The innermost brackets represent the number 5, since

$$\begin{aligned} 5^{32} &= \quad 232\ 83064\ 36538\ 69628\ 90625 \\ 24! &= \ 6204\ 48401\ 73323\ 94393\ 60000 \\ 6^{32} &= 79586\ 61109\ 94640\ 08843\ 91936 \end{aligned}$$

and hence $[\surd\surd\surd\surd\surd 24!] = 5$. The next level of brackets obviously gives 10, since $10 = [\surd 5!]$. Proceeding similarly, we can evaluate the kth brackets to obtain the integer a_k, where

$$a_k^{2^t} < a_{k-1}! < (a_k + 1)^{2^t}$$

The author would like to suggest that solutions of cyclic differential equations should be listed like regular integrals. This would make the student happier, the textbook more economic, and the authors of those texts more and more bewildered.

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$$a_k^{2^t} < a_{k-1}! < (a_k+1)^{2^t}$$

for some appropriate integer t. The details are:

k	a_k	t	$a_k^{2^t}$	$a_{k-1}!$	$(a_k+1)^{2^t}$
3	1904	1	36 25216	36 28800	36 29025
4	442	11	$.67\times 10^{5417}$	$.42\times 10^{5419}$	$.68\times 10^{5419}$
5	6673	8	$.1062\times 10^{979}$	$.1097\times 10^{979}$	$.1104\times 10^{979}$
6	577	13	$.4\times 10^{22619}$	$.9\times 10^{22623}$	$.5\times 10^{22625}$
7	422	9	$.14\times 10^{1344}$	$.25\times 10^{1344}$	$.49\times 10^{1344}$
8	64	9	$.58\times 10^{924}$	$.21\times 10^{926}$	$.16\times 10^{928}$

With such large numbers, of course, logarithms are used for the calculations. Logarithms of factorials up to 1200! are given to 18 places in J. Peters, *Ten Place Logarithm Table*, Ungar, New York, 1957, vol. 1. The logarithms of larger factorials are easily computed by means of Stirling's approximation.

Notice that $[\sqrt{[x]}]=[\sqrt{x}]$, so the bracket operation is needed only around the entire result and before factorials are taken. Since $3=[\sqrt{10}]$, 3 can be represented with one four, hence also $3!=6$, and so on. It seems a plausible conjecture that all positive integers possess such a representation, but this fact (if true) seems to be tied up with very deep properties of the integers.

The referee has suggested a stronger conjecture, that a representation may be found in which all factorial operations precede all square root operations; and moreover, if the greatest integer function is not used, an arbitrary positive real number can probably be approximated as closely as desired in this manner. If N_k is the number $(\cdots((4!)!)!\cdots)!$ with k factorial operations, this conjecture is equivalent to the statement that the values of

$$\log_2 N_k/2^{[\log_2(\log_2 N_k)]},\ k = 1, 2, 3, \cdots$$

are dense in the interval (1, 2).

A computer program was written to see what small numbers can be represented in a fashion similar to our formula for 64. For all x, y less than 1000 it was determined whether or not

$$x = [\sqrt[2^u]{[\sqrt[2^v]{y!}]!}]$$

for some integers u and v, provided all of the calculations could be carried out without numbers becoming too large, and provided it could be guaranteed that round-off error would have absolutely no effect. If we write $y\to x$ provided this relationship holds, we can write $y\to\to x$ meaning $y\to y_1\to y_2\to\cdots\to y_k=x$, for some $y_1, y_2, \cdots$; and then the number n can be represented by one 4 if $4\to\to n$. The results of the computer run were that $4\to\to n$ for all $n<208$. The fact that 208 was not obtained is probably due to the limited search carried out on the computer; however, it is interesting that all numbers $n\leq 200$ can be written as $4\to n_1\to n_2\to\cdots\to n$ where each n_i is at most 1000.

A further improvement can be made by using negation in connection with the other operations, since $-[-x]$ is the least integer greater than or equal to x. This possibility has not been fully explored. For example, it can be used to extend the list of numbers represented in a simple way by four 1's to at least 37, using

$$35 = -[\sqrt{(11+1)}/-.1], \quad 36 = (\sqrt{(1/.\dot{1})})!(\sqrt{(1/.\dot{1})})!, \quad 37 = 1 - [-\sqrt{(11)}]/.\dot{1},$$

and the list of representations of 1 through 34 in [2]. The latter article contains an excellent bibliography.

References

1. J. A. Tierney, E 631, Amer. Math. Monthly, 52 (1945) 219.
2. M. Bicknell and V. E. Hoggatt, 64 ways to write 64 using four 4's, Recreational Mathematics Magazine, 14 (1964) 13–15.

SIMULTANEOUS SOLUTION OF THE THREE ANCIENT PROBLEMS

JOYCE CURRIE LITTLE, Goucher College, Baltimore, Maryland, and
V. C. HARRIS, San Diego State College

In this paper we exhibit a combination of two sine curves and a circle which will solve the problems of trisecting an angle, squaring the circle, and duplicating the cube. It is well known that the solution of these problems is impossible using only euclidean tools and hence that curves not constructible by ruler and compass must be used in getting exact solutions. Our curves have the special advantage of having to be drawn only once for solving the problems simultaneously. The curves are $f_1(\theta)=3\sin\theta$, $f_2(\theta)=2+\sin 3\theta$, and a circle of radius 3 on the same coordinate axis with center O' at $(-3, 0)$. The position is as indicated in Figure 1, and the methods are as follows.

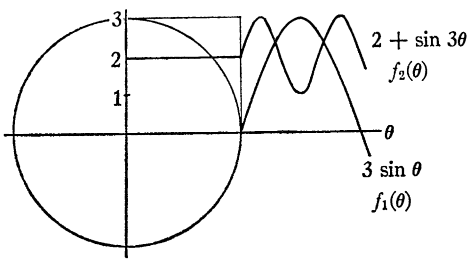

FIG. 1.

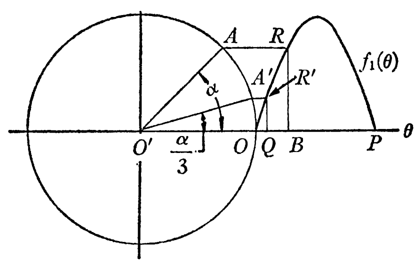

FIG. 2.

To use the curves to trisect an arbitrary angle, first construct an angle equal to the given angle $\alpha<\pi/2$ in standard position with its initial side along the θ-axis and with vertex at the center O' of the circle. (See Fig. 2.) Let the terminal side cut the circle in A. From point A, draw a parallel to the θ-axis, cutting the curve $f_1(\theta)$ in R. Draw the ordinate through R cutting the θ-axis at B. The length OB is α. Trisect the line segment α and let Q be the point of trisection nearer the origin. The abscissa of Q is $\alpha/3$. At Q, construct the ordinate to the curve $f_1(\theta)$, cutting it in R'. At R', draw a parallel to the θ-axis cutting the circle in A'. Draw $O'A'$. The angle labelled $A'O'O$ is $\alpha/3$ and is the solution of the problem.

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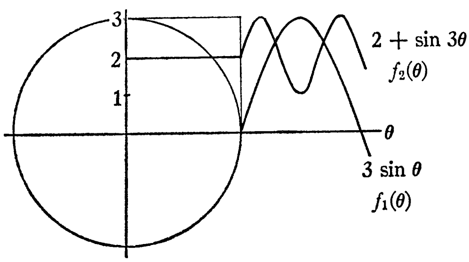

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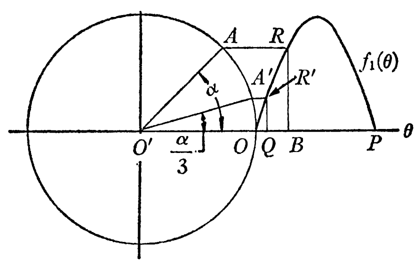

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To use the curves to trisect an arbitrary angle, first construct an angle equal to the given angle $\alpha<\pi/2$ in standard position with its initial side along the θ-axis and with vertex at the center O' of the circle. (See Fig. 2.) Let the terminal side cut the circle in A. From point A, draw a parallel to the θ-axis, cutting the curve $f_1(\theta)$ in R. Draw the ordinate through R cutting the θ-axis at B. The length OB is α. Trisect the line segment α and let Q be the point of trisection nearer the origin. The abscissa of Q is $\alpha/3$. At Q, construct the ordinate to the curve $f_1(\theta)$, cutting it in R'. At R', draw a parallel to the θ-axis cutting the circle in A'. Draw $O'A'$. The angle labelled $A'O'O$ is $\alpha/3$ and is the solution of the problem.

The curves give a solution of the quadrature problem. Since the curve $f_1(\theta)=3\sin\theta$ has period 2π, the length OP (see Fig. 2) is equal to π. With length π given, $\sqrt{\pi}$ may then be constructed, and construction of the square with side $\sqrt{\pi}$ and area π completes the solution.

Also the curves may be used to solve the duplication problem. The simultaneous solution of $f_1(\theta)$ and $f_2(\theta)$ gives $3\sin\theta=\sin 3\theta+2$. Replacing $\sin 3\theta$ by $3\sin\theta-4\sin^3\theta$, we get $\sin^3\theta=1/2$. Thus the ordinate of the intersection of $f_1(\theta)$ and $f_2(\theta)$ has length $3(1/\sqrt[3]{2})$. First $1/\sqrt[3]{2}$ and then $\sqrt[3]{2}$ may be constructed; this is the side of the desired cube.

Some higher plane curves provide solutions to two of the problems. The quadratrix of Hippias suffices for the solution of both trisection and quadrature; the spiral of Archimedes is sufficient for the same two; and the conchoid of Nicomedes can be used to solve both trisection and duplication. Although only one quadratrix or one spiral is needed to solve both problems, two distinct conchoids are necessary. Indeed a different conchoid is needed for trisecting each different angle. It may be of interest to note that no one of these curves mentioned is known to solve all three problems—in fact, no one of these is known to solve both quadrature and duplication. Although the article on quadratrix appearing in the eleventh Edition of the *Encyclopaedia Britannica* states that the quadratrix solves all three problems, this statement has been removed from subsequent editions and is thought to be an error.

The multisection of an arbitrary angle is as easily accomplished using these curves as trisection. Also certain other generalizations requiring $\sqrt[3]{n}$ or $\sqrt[n]{2}$ can be solved using somewhat more complicated trigonometric curves.

RIEMANN INTEGRALS AS MAPPINGS IN ELEMENTARY APPLICATIONS

LYLE E. PURSELL, Grinnell College

Introduction. In most applications of the definite integral in beginning calculus we wish to determine the numerical measure $M(a, b; f)$ of some physical or geometrical quantity associated with a function f defined on some interval $a\leqq x\leqq b$. For example, we might want to determine the work $W(a, b; f)$ done when a particle acted on by a force f is moved along the x-axis from a to b or we might want to determine the volume $V(a, b; f)$ generated when the region, $a\leqq x\leqq b$, $0\leqq y\leqq f(x)$, is revolved about the y-axis. Starting with an algebraic formula, inherited from pre-calculus mathematics or physics, which gives M for the case f is a constant function and a set of additional assumptions about this measure (generally: M is additive in the interval variables, i.e., $M(a, b; f)+M(b, c; f)=M(a, c; f)$, and M is order preserving in the function variable, i.e., if $f\leqq g$, then $M(a, b; f)\leqq M(a, b; g)$), we then show for an arbitrary partition of the interval that $M(a, b; f)$ may be approximated by a Riemann sum or upper and lower Darboux sums. Passing to the limit, we obtain the desired

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In this paper we show that in a wide variety of elementary applications the integral formula may be read off directly from the formula for the constant case. It is not necessary to go through the process of justifying Riemann or Darboux sums for each type of application.

We consider a real-valued mapping T defined on the cartesian product of the set of all bounded closed subintervals of some interval I (not necessarily closed or bounded) and any set F of real-valued continuous functions from I into an interval J (not necessarily closed or bounded) such that F contains all constant-valued functions on I with values in J (i.e., F is any set such that $\mathbf{J}\subseteq F\subseteq C(I, J)$ where $\mathbf{J}$ is the set of all constant functions on I with values in J and $C(I, J)$ is the set of all continuous functions defined on I with values in J). We show that if the mapping T is a finitely additive set function on the set of bounded, closed subintervals of I and order-preserving on F, then T is uniquely determined by its values on the set of constant functions. Specifically (Theorem I) if ϕ is a given monotone nondecreasing continuous function defined on J and $T(a, b; \mathbf{c})=\phi(c)(b-a)$ for all intervals $a\leqq x\leqq b$ in I and all constant functions $\mathbf{c}$ ($\mathbf{c}(x)=c$ for all x in I) in F, then $T(a, b; f)=\int_a^b\phi(f(x))dx$ for all functions f in F and (Theorem II) if w is a monotone nondecreasing, continuously differentiable function defined on I and $T(a, b, \mathbf{c})=c(w(b)-w(a))$, for all subintervals $a\leqq x\leqq b$ in I and all constants $\mathbf{c}$ in F, then $T(a, b; f)=\int_a^b f(x)w'(x)dx$ for all f in F.

We discuss several applications, illustrating how these theorems enable us legitimately to read off the definite interval formula directly from the formula for the constant case without first setting up and justifying an approximating Riemann sum. In each "type" of application the function ϕ or w as the case may be is determined by the formula for the constant case and is the same function for all cases of the type. For example in the two types mentioned above, $\phi(x)=x$ for all x for the work problem and $w(x)=\pi x^2$ for the problem of the volume of a solid of revolution.

We also discuss briefly extensions of Theorem I to multiple integrals and of Theorem II to Riemann-Stieltjes integrals.

1. Bibliographic Notes. The specific theorems and examples given here were worked out independently by the author; but the idea of treating a definite integral as a mapping is not original of course. In the "Riesz representation theorem" (1909) Stieltjes integrals are treated as linear functionals L defined on a suitable algebra of functions such that $L(f)\geqq 0$ for all nonnegative functions in the algebra. This theorem and some of its more recent modifications and generalizations are discussed by Edwin Hewitt in "The role of compactness in analysis," *The American Mathematical Monthly*, 67 (1960), 499–516, and in the kinescope, "What is an Integral?," shown at the Forty-fourth Summer meeting of the Mathematical Association of America at Boulder, Colorado on August 26, 1963. In *Delta-Epsilon*, III (January, 1963), p. 47 (a student-staff publication of Carleton College) Frank L. Wolf presents a conjecture similar to Theorem I as a problem for investigation by students. Professor Wolf only considers the case for which ϕ is the identity function, I is the whole real line R, and F is the set of all real-valued continuous functions on R, but his formulation of the problem suggests that the set-additive hypothesis for $T(a, b; f)$ might be replaced by the assumption that $T(a, a; f) = 0$ and $T(a, b; f)$ is a continuous function of the variables a and b. Readers interested in an unconventional treatment of definite integrals should also see D. E. Richmond's paper, "Calculus—A New Look" in the Classroom Notes of *The American Mathematical Monthly*, 70 (1963), 415–423.

The use of bold-faced type to denote constant-valued functions is adopted from *Rings of Continuous Functions* by Leonard Gillman and Meyer Jerison, Van Nostrand, 1960.

2. Integrals of the form $\int_a^b \phi(f(x))dx$.

(2.1) THEOREM I. *Let ϕ be a monotone nondecreasing, continuous function defined on an interval J (not necessarily closed or bounded). Let F be a nonempty set of continuous functions f defined on an interval I (not necessarily closed or bounded) such that the range of f is in J and if k is in J, then the constant valued function* $\mathbf{k}$, *defined on I, is in F. Let $S(I, F)$ be the set of all triples of the form $(a, b; f)$ such that a and b are in I, $a<b$, and f is in F. If T is a mapping from $S(I, F)$ into the real field such that:*

(2.11) *for a fixed f, T is a finitely additive set function defined on the set of closed subintervals of I, i.e., if $a<b<c$, then $T(a, c; f) = T(a, b; f) + T(b, c; f)$;*

(2.12) *for a given subinterval $a \leqq x \leqq b$, T is an order preserving mapping defined on the set of functions in F restricted to $a \leqq x \leqq b$, i.e., if $f(x) \leqq g(x)$ for all x in $a \leqq x \leqq b$, then $T(a, b; f) \leqq T(a, b; g)$;*

(2.13) *for a constant valued function* $\mathbf{c}$, $T(a, b; \mathbf{c}) = \phi(c)(b-a)$,

then

(2.14) $T(a, b; f) = \int_a^b \phi(f(x))dx$ *for all* $(a, b; f)$ *in* $S(I, F)$.

Proof. Obviously the mapping $T(a, b; f) = \int_a^b \phi(f(x))dx$ does satisfy the hypothesis. Since such a mapping exists, we only have to establish its uniqueness.

For an arbitrary partition $a = x_0 \leqq x_1 \leqq \cdots \leqq x_i \leqq \cdots \leqq x_n = b$ we have $T(a, b; f) = \sum_i T(x_{i-1}, x_i; f)$ from (2.11). If we let m_i denote the minimum value of $f(t)$ and M_i denote the maximum value of $f(t)$ on the interval $x_{i-1} \leqq t \leqq x_i$, then by (2.13) and (2.12):

$$\begin{aligned} \phi(m_i)(x_i - x_{i-1}) &= T(x_{i-1}, x_i; \mathbf{m}_i) \\ &\leqq T(x_{i-1}, x_i; f) \\ &\leqq T(x_{i-1}, x_i; \mathbf{M}_i) = \phi(M_i)(x_i - x_{i-1}) \end{aligned}$$

for all i. Hence

$$\sum_i \phi(m_i)(x_i - x_{i-1}) \leqq \sum_i T(x_{i-1}, x_i; f) \leqq \sum_i \phi(M_i)(x_i - x_{i-1})$$

for all partitions of the interval $a \leqq x \leqq b$. But the bounding sums in this inequality converge to the Riemann integral $\int_a^b \phi(f(x))dx$ as the norm of the subdivision goes to zero. Therefore

$$T(a, b; f) = \sum_i T(x_{i-1}, x_i; f) = \int_a^b \phi(f(x))dx.$$

The assumption that ϕ be nondecreasing is necessary although it does not appear explicitly in the proof. If ϕ is not nondecreasing then the mapping $T(a, b; f) = \int_a^b \phi(f(x))dx$ does not satisfy (2.12). If we assume that ϕ is monotone nonincreasing, then we must replace (2.12) by

$$(2.12') \qquad T(a, b; f) \geqq T(a, b; g) \text{ if } f(x) \leqq g(x) \text{ for all } x \text{ in } a \leqq x \leqq b,$$

in order to obtain (2.14).

For continuous functions one may prove Theorem I in a different way by first proving that

$$(2.2) \qquad T(a, b; f) = \phi(f(t))(b - a) \quad \text{for some } t \text{ in } a \leqq t \leqq b.$$

This result, which is analogous to the Mean Value Theorem for integrals, is obtained by the use of the Intermediate Value Theorem from the inequality

$$\phi(m)(b - a) = T(a, b; \boldsymbol{\phi}(\mathbf{m})) \leqq T(a, b; f) \leqq T(a, b; \boldsymbol{\phi}(\mathbf{M})) = \phi(M)(b - a)$$

in which m and M are the minimum and maximum values respectively of the function f on the interval $a \leqq x \leqq b$. (This inequality is valid because ϕ is monotone nondecreasing and T is order-preserving.) Now from (2.2) and (2.11) we have, for $h > 0$,

$$T(a, x + h; f) - T(a, x; f) = T(x, x + h; f) = \phi(f(t)) \cdot h$$

for some t such that $x \leqq t \leqq x + h$ and, for $h < 0$,

$$T(a, x + h; f) - T(a, x; f) = - T(x + h, x; f) = - \phi(f(t)) \cdot (-h)$$

for some t such that $x + h \leqq t \leqq x$. Since $\phi \circ f$ is continuous, then

$$\begin{aligned} dT(a, x; f)/dx &= \lim_{h \to 0} (T(a, x + h; f) - T(a, x; f))/h \\ &= \lim_{h \to 0} \phi((f(t)) = \phi(f(x)). \end{aligned}$$

Since $\lim_{x \to a} T(a, x; f) = 0$, it then follows from the Fundamental Theorem of Calculus that $T(a, b; f) = \int_a^b \phi(f(x))dx$.

3. Applications of Theorem I.

(3.1) *Work.* Let $W(a, b; f)$ denote the work done on a particle as it is moved along the x-axis from a to b by a continuous force function f. For a constant force function **c** we have the familiar formula from elementary physics

$$(3.11) \qquad W(a, b; \mathbf{c}) = c(b - a).$$

In extending our elementary definition of work to continuous force functions which are not necessarily constant, we wish to retain the properties: (i) a greater force does greater work, i.e., (3.12) if $f(x) \leqq g(x)$ for all x in $a \leqq x \leqq b$, then $W(a, b; f) \leqq W(a, b; g)$ and (ii) work is "additive," i.e.,

(3.13) $\quad$ if $a < b < c$, then $W(a, c; f) = W(a, b; f) + W(b, c; f)$.

From (3.11) we have $\phi(y) = y$ for all y and it is easy to see that all of the hypothesis of Theorem I is satisfied. Hence $W(a, b; f) = \int_a^b f(x)dx$.

(3.2) *Area in polar coordinates.* Let $A(a, b; f)$ denote the area bounded by the rays $\theta = a$, $\theta = b$, and the graph of $r = f(\theta)$ where $0 \leqq a < b \leqq 2\pi$ and f is a nonnegative continuous function defined on $0 \leqq \theta \leqq 2\pi$. We assume that area is a nonnegative additive set function, that congruent regions have equal areas, and that the area of a circular disc of radius r is given by the standard formula πr^2. From these assumptions we obtain:

(3.21) $\quad A(a, c; f) = A(a, b; f) + A(b, c; f)$ for $a < b < c$,

(3.22) $\quad A(a, b; f) \leqq A(a, b; g)$ if $f(\theta) \leqq g(\theta)$ for all θ in $a \leqq \theta \leqq b$,

and

(3.23) $\quad A(a, b; \mathbf{c}) = \frac{1}{2} c^2(b - a)$ for all positive constants $\mathbf{c}$.

According to (3.23) we have to take $\phi(r) = \frac{1}{2}r^2$, $0 \leqq r$, and we see that ϕ satisfies the conditions of Theorem I and, in particular, $A(a, b; \mathbf{c}) = \phi(c)(b - a)$. Hence

$$A(a, b; f) = \int_a^b \phi(f(\theta))d\theta = \int_a^b \frac{1}{2} r^2 d\theta,$$

where $r = f(\theta)$.

(3.3) *Arc Length.* The conventional definition of arc length as the limit of the length of approximating polygonal paths does not readily lend itself to our treatment. Instead we assume that we may associate with each continuously differentiable function f defined on $a \leqq x \leqq b$ a number $L(a, b; f)$ which we call the arc length of the graph of f. We further assume:

(3.31) arc length is additive, i.e., $L(a, c; f) = L(a, b; f) + L(b, c; f)$ if $a < b < c$,

(3.32) the steeper of two graphs over a given interval will have the greater arc length, i.e., $L(a, b; f) \leqq L(a, b; g)$ if $|f'(x)| \leqq |g'(x)|$ for all x in $a \leqq x \leqq b$,

(3.33) for a linear function $f(x) = mx + y_0$, the arc length is given by the familiar distance formula of elementary analytic geometry, i.e., $L(a, b; mx + y_0) = (1 + m^2)^{1/2}(b - a)$.

The arc length of a curve depends on its shape rather than its position. Also two curves have the same arc length if one is a reflection of the other. Hence the arc length of the graph of f is determined by the absolute value of its derivative $|f'|$; i.e.,

$$L(a, b; f) = L(a, b; g) \quad \text{if } |f'(x)| = |g'(x)| \quad \text{for all } x \text{ in } a \leqq x \leqq b.$$

(Of course this property also follows from (3.32).) Consequently, we may set $L(a, b; f) = T(a, b; |f'|)$; then (3.33) becomes

$$T(a, b; |\mathbf{m}|) = (1 + m^2)^{1/2}(b - a).$$

Therefore $\phi(|m|) = (1 + m^2)^{1/2}$. Taking $F(I)$ as the set of all nonnegative continuous functions $|f'|$ defined on any interval I, one can show that the hypothesis of Theorem I is satisfied, hence:

$$L(a, b; f) = \int_a^b \phi(|f'(x)|)dx = \int_a^b (1 + (f'(x))^2)^{1/2}dx$$

for all continuously differentiable functions f.

4. Integral formulas with a weight function. If the rectangular region $\{(x, y): 0 \leqq a \leqq x \leqq b,\ 0 \leqq y \leqq c\}$ is revolved about the y-axis, then the formula for the volume generated is of the form $c(w(b) - w(a))$ where $w(x) = \pi x^2$. Since w is not the identity function this formula is not of the type obtained in sections 2 and 3. As

$$c(w(b) - w(a)) = \int_a^b \mathbf{c}\, dw(x),$$

the results of the preceding section suggest that the formula for the volume generated by revolving the region $\{(x, y): 0 \leqq a \leqq x \leqq b,\ 0 \leqq y \leqq f(x)\}$ about the y-axis is $\int_a^b f(x)d(\pi x^2)$. (This corresponds to the "cylindrical shell method".) Extending this analogy we see that it appears that if we replace hypothesis (2.13) of Theorem I by $T(a, b; \mathbf{c}) = c(w(b) - w(a))$ then we will obtain

$$T(a, b; f) = \int_a^b f(x)dw(x).$$

To maintain the order-preserving property (2.12) we assume that w is monotone nondecreasing on the interval I.

(4.1) THEOREM II. *Let w be a monotone nondecreasing, continuously differentiable function defined on an interval I. Let $F(I)$ and $S(I, F)$ be defined as in Theorem I. If T is a mapping from $S(I, F)$ into the real field such that:*

(4.11) $T(a, c; f) = T(a, b; f) + T(b, c; f)$ *if* $a < b < c$,

(4.12) $T(a, b; f) \leqq T(a, b; g)$ *if* $f(x) \leqq g(x)$ *for all x in* $a \leqq x \leqq b$,

(4.13) *for a constant valued function* $\mathbf{c}$, $T(a, b; \mathbf{c}) = c(w(b) - w(a))$,

then

$$T(a, b; f) = \int^b f(x)w'(x)dx. \tag{4.14}$$

Proof. Consider a partition: $a = x_0 \leqq x_1 \leqq \cdots \leqq x_{i-1} \leqq x_i \leqq \cdots \leqq x_n = b$. Using the Intermediate Value Theorem one can show that there is a number t_i^* such that $x_{i-1} \leqq t_i^* \leqq x_i$ for which

$$T(x_{i-1}, x_i; f) = f(t_i^*)(w(x_i) - w(x_{i-1})). \tag{4.15}$$

(cf. the standard proof of the Mean Value Theorem of Integral Calculus and the two proofs of Theorem I.) By the Mean Value Theorem of Differential Calculus there is a number t_i^{**} such that $x_{i-1} \leqq t_i^{**} \leqq x_i$ and $w(x_i) - w(x_{i-1}) = w'(t_i^{**})(x_i - x_{i-1})$. Hence

$$T(a, b; f) = \sum_i T(x_{i-1}, x_i; f) = \sum_i f(t_i^*)w'(t_i^{**})(x_i - x_{i-1}).$$

According to Duhamel's Theorem the limit of the latter sum is $\int_a^b f(x)w'(x)dx$ as the norm of the partition goes to zero.

We can now establish the formula for the volume of a solid of revolution by the cylindrical shell method. If $V(a, b; f)$ is the volume of a solid generated by revolving the region $\{(x, y): 0 \leqq a \leqq x \leqq b,\ 0 \leqq y \leqq f(x)\}$ about the y-axis, then we have:

$$V(a, c; f) = V(a, b; f) + V(b, c; f) \quad \text{if} \quad a < b < c,$$

$$V(a, b; f) \leqq V(a, b; g) \quad \text{if} \quad f(x) \leqq g(x) \quad \text{for all } x \text{ in } a \leqq x \leqq b, \quad \text{and}$$

$$V(a, b; c) = c(\pi b^2 - \pi a^2).$$

Hence $V(a, b; f) = T(a, b; f)$ if $w(x) = \pi x^2$. Therefore

$$V(a, b; f) = \int_a^b f(x) \cdot \frac{d}{dx} (\pi x^2) dx = 2\pi \int_a^b x f(x) dx.$$

5. The Riemann-Stieltjes integral. If the weight function w of Theorem II is monotone, nondecreasing but is not necessarily differentiable, then in place of (4.14) we obtain:

$$T(a, b; f) = \int_a^b f(x) dw(x), \tag{5.1}$$

the Riemann-Stieltjes integral of f with respect to w from a to b. This is readily established by summing both sides of equation (4.15) in the proof of Theorem II with respect to i, then taking the limit of this sum as the norm of the partition goes to zero.

6. Double integrals. Theorem I can be extended to multiple integrals. Suppose we are given a region R, a set $F(R)$ of continuous functions defined on R which is sufficiently rich in constants, and a set $\mathcal{R}$ of subregions, r, of R such that R can be partitioned into nonoverlapping subregions in $\mathcal{R}$ of arbitrarily small diameter (more precisely: for each $\epsilon > 0$, there is a subset $\mathcal{P}$ of $\mathcal{R}$ which is a partition of R such that the diameter of every region in $\mathcal{P}$ is less than ϵ). If T is a mapping from $\mathcal{R} \times F(R)$ into the real field which is finitely additive with respect to r, is order preserving with respect to f and is such that

(6.1) $T(r, c) = c\mu(r)$, where μ is the measure of r, then

$$T(r, f) = \int_r f d\mu. \tag{6.2}$$

Further if ϕ is a continuous, monotone nondecreasing function whose domain contains the ranges of all functions in $F(R)$ and if

(6.3) $T(r, c) = \phi(c)\mu(r)$, then

$$T(r, f) = \int_r \phi \circ f d\mu. \tag{6.4}$$

The proofs are analogous to the proof of Theorem I.

I wish to express my appreciation to Professor C. N. Wollan of Purdue University, Mr. Charles J. Cook of Grinnell College, and the referee for their suggestions for improving this paper.

A NOTE ON STIRLING NUMBERS OF THE FIRST KIND

L. CARLITZ, Duke University

The Stirling number $s(n, k)$ is usually defined by means of

$$(1) \qquad (x)_n = x(x-1)\cdots(x-n+1) = \sum_{r=0}^{n} s(n, r)x^r.$$

It is obvious that the $s(n, r)$ are all integers. If p is an arbitrary prime and n is fixed we let $\omega(n) = \omega_p(n)$ denote the number of $s(n, r)$ that are not divisible by p.

It is very easy to determine $\omega_2(n)$. Indeed since

$$(x)_{2n} \equiv x^n(x-1)^n \equiv \sum_{r=0}^{n} \binom{n}{r} x^{n+r} \pmod 2,$$

it follows from (1) that

$$\begin{cases} s(2n, n+r) \equiv \binom{n}{r} \pmod 2 & (0 \leqq r \leqq n) \\ s(2n, r) \equiv 0 \pmod 2 & (0 \leqq r < n). \end{cases}$$

If we put

$$(2) \qquad n = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_k} \qquad (0 \leqq e_1 < e_2 < \cdots < e_k),$$

it is familiar that the number of odd binomial coefficients $\binom{n}{r}$, where n is fixed, is equal to 2^k. Similarly from

$$(x)_{2n+1} \equiv x^{n+1}(x-1)^n \equiv \sum_{r=0}^{n} \binom{n}{r} x^{n+r+1} \pmod 2$$

it follows that

$$\begin{cases} s(2n+1, n+r+1) \equiv \binom{n}{r} \pmod 2 & (0 \leqq r \leqq n) \\ s(2n+1, r) \equiv 0 \pmod 2 & (0 \leqq r \leqq n). \end{cases}$$

We have therefore

$$(3) \qquad \omega_2(2n) = \omega_2(2n+1) = 2^k,$$

where k is defined by (2).

For an arbitrary prime p we have first

$$(x)_{pn} \equiv x^n(x^{p-1}-1)^n \equiv \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} x^{n+r(p-1)} \pmod p.$$

Thus by (1)

$$(4) \qquad s(pn, n+r(p-1)) \equiv (-1)^{n-r} \binom{n}{r} \pmod p \qquad (0 \leqq r \leqq n),$$

while $s(pn, t) \equiv 0 \pmod{p}$ for all other values of t not covered by (4). If we put

(5) $$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqq n_j < p)$$

then it is well known [1] that the binomial coefficient $\binom{n}{r}$ is prime to p if and only if $r = r_0 + r_1 p + \cdots + r_k p^k$ and $0 \leqq r_j \leqq n_j (0 \leqq j \leqq k)$. Thus there are $(1+n_0)(1+n_1) \cdots (1+n_k)$ values of r for which $\binom{n}{r}$ is prime to p. It follows at once from (4) that

(6) $$\omega_p(pn) = (1 + n_0)(1 + n_1) \cdots (1 + n_k).$$

Since $(x)_{pn+1} \equiv x^{n+1}(x^{p-1} - 1)^n \pmod{p}$, we get

$$s(pn + 1, n + r(p - 1) + 1) \equiv (-1)^{n-r}\binom{n}{r} \pmod{p}$$

and therefore

(7) $$\omega_p(pn + 1) = \omega_p(pn).$$

Generally for $1 \leqq t < p$ we have $\omega_p(pn+t) \equiv x^n(x^{p-1} - 1)^n (x)_t \pmod{p}$, so that

$$\omega_p(pn + t) \equiv \sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} x^{n+r(p-1)} \cdot \sum_{j=0}^{t} s(t, j)x^j \pmod{p}.$$

Since for $t \geqq 1$, $s(t, 0) = 0$, it follows that

(8) $$s(pn + t, n + r(p - 1) + j) \equiv (-1)^{n-r}\binom{n}{r} s(t, j) \pmod{p}$$

and that $s(pn+t, u) \equiv 0 \pmod{p}$ for all other values of u not covered by (8). Note that the sums $n + r(p-1) + j$, $(0 \leqq r \leqq n;\ 1 \leqq j \leqq t)$ are distinct. We have therefore

(9) $$\omega_p(pn + t) = \omega_p(pn)\omega_p(t) \qquad (1 \leqq t < p)$$

with $\omega_p(pn)$ determined by (6).

Thus the evaluation of $\omega_p(n)$ is reduced to the evaluation of $\omega_p(t)$ where $t < p$. It is obvious that $\omega_p(t) \leqq t$, $(1 \leqq t < p)$. It may happen, however, that $\omega_p(t) < t$. For example from a table of values of $s(n, r)$ [2, p. 48] we find that

$$\omega_{11}(4) = 3,$$

$$\omega_7(5) = 4,$$

$$\omega_{17}(6) = \omega_{137}(6) = 5,$$

$$\omega_{29}(7) = 6,$$

$$\omega_{11}(8) = \omega_{23}(8) = \omega_{67}(8) = \omega_{967}(8) = 7.$$

Moreover we can assert that

$$\begin{aligned} \omega_3(3n+2) &= 2\omega_3(3n), \\ \omega_5(5n+t) &= t\omega_5(5n) \qquad (t = 2, 3, 4), \\ \omega_7(7n+t) &= t\omega_7(7n) \qquad (t = 2, 3, 4, 6), \\ \omega_7(7n+5) &= 4\omega_7(7n), \\ \omega_{11}(11n+t) &= t\omega_{11}(11n) \qquad (t = 2, 3, 5, 6, 7, 9, 10), \\ \omega_{11}(11n+4) &= 3\omega_{11}(11n), \\ \omega_{11}(11n+8) &= 7\omega_{11}(11n). \end{aligned}$$

It is easy to show that

$$\omega_p(p-1) = p-1. \tag{10}$$

Indeed

$$(x)_{p-1} = \frac{(x)_p}{x-p+1} \equiv \frac{x^p - x}{x+1} \equiv (x+1)^{p-1} - 1 \equiv \sum_{r=0}^{p-2} (-1)^r x^{p-r-1} \pmod p$$

and (10) follows at once.

In the next place

$$(x)_{p-2} = \frac{(x)_p}{(x-p+1)(x-p+2)} \equiv \frac{x^p - x}{(x+1)(x+2)} \pmod p.$$

Thus if we put $(x)_{p-2} = \sum_{r=0}^{p-3} c_r x^{p-r-2}$, we have $c_r + 2c_{r-1} \equiv (-1)^r$. This yields $c_r = (-1)^r(2^{r+1}-1)$. Hence,

$$\omega_p(p-2) = p-2 \tag{11}$$

if 2 is a primitive root of p, but

$$\omega_p(p-2) < p-2 \tag{12}$$

if 2 is not a primitive root of p. In particular

$$\omega_p(p-2) < p-2 \qquad (p \equiv \pm 1 \pmod 8) \tag{13}$$

for, in this case, 2 is a quadratic residue of p. More precisely if 2 belongs to the exponent $k \pmod p$ then

$$\omega_p(p-2) = p - 1 - \frac{p-1}{k} = \frac{(k-1)(p-1)}{k}. \tag{14}$$

If we put $(x)_{p-3} = \sum_{r=0}^{p-4} d_r x^{p-r-3}$, it is evident that

$$d_r + 3d_{r-1} = (-1)^r(2^{r+1}-1).$$

It follows that

$$d_r \equiv \frac{(-1)^r}{2}(3^{r+2} - 2\cdot 2^{r+2} + 1) \pmod p. \tag{15}$$

Thus, for example, if both 2 and 3 are quadratic residues of p, we have

$$\omega_p(p-3) < p-3. \tag{16}$$

However, it is not clear how to obtain an exact result like (14) for $\omega_p(p-3)$.

The congruence (15) can be generalized in an obvious way. Indeed if

$$(x)_{p-k} = \sum_{r=0}^{p-k-1} c_r^{(k)} x^{p-k-r},$$

then

$$c_r^{(k)} \equiv \frac{(-1)^r}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} (j+1)^{k+r} \pmod{p}.$$

In particular, it is clear from the above discussion, that there exist infinitely many pairs p, t with $t<p$ such that $\omega_p(t)<t$.

Supported in part by NSF grant GP-1593.

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2. J. Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.

ANSWERS

A 344. Let A' denote the image of A with respect to O, the center of the sphere. Then $OA+OB>AB$ (in general). Whence, $\overline{OA}+\overline{OB}>\overline{AB}$. But

$$\overline{OA} = \overline{OB} = \int_0^r r\cdot 4\pi r^2 dr \div \int_0^r 4\pi r^2 dr = 3r/4$$

Also $\overline{AB}+\overline{AB'}>2\overline{OA}$. Since $\overline{AB}=\overline{AB'}$ the inequalities follows.

A 345. $(7^{\sqrt{5}})^{\sqrt{35}}=7^{5\sqrt{7}}=(16{,}807)^{\sqrt{7}}>(15{,}625)^{\sqrt{7}}=5^{6\sqrt{7}}=5^{\sqrt{252}}>5^{\sqrt{245}}=5^{7\sqrt{5}}=(5^{\sqrt{7}})^{\sqrt{35}}$. Hence $7^{\sqrt{5}}>5^{\sqrt{7}}$.

A 346. No one of the digits is zero. $(4)(6)(8)=192$, and $(2)(4)(6)=48$. $\sqrt[3]{87}=4.4+$ and $\sqrt[3]{88}=4.4+$. Therefore, the product is $(442)(444)(446)=87526608$.

A 347. If $(WXYZ\cdots U)$ are the digits of an integer, it can be shown that $(WXYZ\cdots U)^2=(W^2+X^2+Y^2+\cdots+U^2)+(2W)X+(2WX)Y+(2WXYZ)T+\cdots+(2WXY\cdots)U$. Thus $(1234)^2=(1000^2+200^2+30^2+4^2)+(2\cdot 1000)200+(2\cdot 1200)30+(2\cdot 1230)4=1{,}522{,}756$.

A 348. If two of the forces are skew, it would be possible to get a nonzero moment about an axis intersecting these two axes. Consequently, these two forces must lie in a plane and intersect (possibly at infinity). Then the third force (by moments) must lie in this plane and be concurrent to the other two.

(Quickies on page 360)

ON NORRIE'S IDENTITY

KENNETH S. WILLIAMS, University of Toronto

The first example expressing a biquadrate as the sum of four biquadrates was given by Norrie (University of St. Andrews 500th Anniversary Memorial vol., Edinburgh, 1911, 89). I give a simple demonstration of this result:

$$442^2 - 272^2 = 170\cdot 714 = 17^2\cdot 420,$$

hence $442^2-3\cdot 17^2=272^2+289\cdot 417=272^2+353^2-64^2$, but $3\cdot 17=2\cdot 26-1$, so

$$442^2 - 2\cdot 26\cdot 17 + 17 = 442^2 - 2\cdot 442 + 17 = 441^2 + 4^2 = 21^4 + 2^4 = 272^2 + 353^2 - 8^4.$$

Hence, $353^2+272^2=2^4+8^4+21^4$, but $353^2-272^2=81\cdot 625=15^4$, so $353^4 =30^4+120^4+272^4+315^4$.

A FIBONACCI-LIKE SEQUENCE OF COMPOSITE NUMBERS

R. L. GRAHAM, Bell Telephone Laboratories, Inc.

Introduction. Let $S(L_0, L_1)=(L_0, L_1, L_2, \cdots)$ be a sequence of integers which satisfy the recurrence

$$L_{n+2} = L_{n+1} + L_n, \qquad n = 0, 1, 2, \cdots.$$

It is clear that the values of L_0 and L_1 determine $S(L_0, L_1)$, e.g., $S(0, 1)$ is just the sequence of Fibonacci numbers. It is not known whether or not infinitely many primes occur in $S(0, 1)$. On the other hand, if there is a prime p which divides both L_0 and L_1, then all the terms of $S(L_0, L_1)$ are divisible by p and in this case it is easily shown that only a finite number of the L_n can be prime. In this paper we exhibit two integers M and N with the following properties:

1. M and N are relatively prime.
2. No term of $S(M, N)$ is a prime number.

Preliminary remarks. Let L_0 and L_1 be arbitrary integers. Denote the nth Fibonacci number by F_n (where F_n is defined for *all* integers n by $F_0=0$, $F_1=1$, $F_{n+2}=F_{n+1}+F_n$, i.e., $F_{-1}=1$, $F_{-2}=-1$, etc.).

For any $m\geqq 0$ we have

$$L_m = 1\cdot L_m + 0\cdot L_{m+1} = F_{-1}L_m + F_0L_{m+1}$$
$$L_{m+1} = 0\cdot L_m + 1\cdot L_{m+1} = F_0L_m + F_1L_{m+1}$$
$$L_{m+2} = 1\cdot L_m + 1\cdot L_{m+1} = F_1L_m + F_2L_{m+1}.$$

Since $(F_nL_m+F_{n+1}L_{m+1})+(F_{n+1}L_m+F_{n+2}L_{m+1})=(F_{n+2}L_m+F_{n+3}L_{m+1})$, then by induction on n, it follows that

(1) $$L_{m+n} = F_{n-1}L_m + F_nL_{m+1}$$

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$$L_{m+1} = 0\cdot L_m + 1\cdot L_{m+1} = F_0L_m + F_1L_{m+1}$$
$$L_{m+2} = 1\cdot L_m + 1\cdot L_{m+1} = F_1L_m + F_2L_{m+1}.$$

Since $(F_nL_m+F_{n+1}L_{m+1})+(F_{n+1}L_m+F_{n+2}L_{m+1})=(F_{n+2}L_m+F_{n+3}L_{m+1})$, then by induction on n, it follows that

$$L_{m+n} = F_{n-1}L_m + F_nL_{m+1} \tag{1}$$

for any m, $n=0, 1, 2, \cdots$. Let p be a prime and let $r(p)$ denote the rank of apparition of p in the F_n, i.e., $r(p)$ is the smallest positive integer r for which $F_r \equiv 0 \pmod{p}$. Then for any $m \geqq 0$, we have by (1)

$$L_{m+r(p)} = F_{r(p)-1}L_m + F_{r(p)}L_{m+1} \equiv F_{r(p)-1}L_m \pmod{p}.$$

Thus,

$$L_m \equiv 0 \Rightarrow L_{m+r(p)} \equiv 0 \Rightarrow L_{m+kr(p)} \equiv 0 \pmod{p}$$

for any integer $k \geqq 0$.

Construction of M and N. Consider the following table:

n	a_n	$r(a_n)$	b_n	n	a_n	$r(a_n)$	b_n
1	2	3	2	10	41	20	10
2	3	4	1	11	53	27	16
3	5	5	1	12	109	27	7
4	7	8	3	13	31	30	24
5	17	9	4	14	2207	32	15
6	11	10	2	15	5779	54	52
7	61	15	3	16	2521	60	60
8	47	16	7	17	1087	64	31
9	19	18	10	18	4481	64	63

The a_n are prime numbers and the corresponding ranks of apparition $r(a_n)$ are easily verified. It is now asserted that every integer belongs to at least one of the arithmetic progressions

$$A_n = \{r(a_n)k + b_n:\ k = 0, \pm 1, \pm 2, \cdots\}$$

for $n=1, 2, \cdots, 18$ (i.e., the A_n form a covering set for the integers). First we see that A_{17}, A_{18}, A_{14}, A_8, A_4, and A_2 cover all the odd integers. Then A_{15}, A_{12}, A_{11}, A_9, A_5, and A_1 cover the remaining integers except those of the form $6n$. Next, A_{13}, A_7, A_6, and A_3 represent all the remaining integers except those of the form $30n$. Finally, since A_{10} and A_{16} cover the integers of the form $30n$, then *all* integers belong to at least one A_i.

Now, notice that if $L_{b_n} \equiv 0 \pmod{a_n}$ then $L_{b_n+kr(a_n)} \equiv 0 \pmod{a_n}$ for $k=0, 1, 2, \cdots$, i.e., $L_x \equiv 0 \pmod{a_n}$ for any x in A_n. Thus, if L_0 and L_1 can be chosen so that $L_{b_n} \equiv 0 \pmod{a_n}$ for $n=1, 2, \cdots, 18$ then *every* term of $S(L_0, L_1)$ is divisible by some a_n. This choice is easily made, for if we take

$$(2) \qquad \begin{aligned} L_0 &\equiv F_{r(a_n)-b_n} \pmod{a_n} \\ L_1 &\equiv F_{r(a_n)-b_n+1} \pmod{a_n} \end{aligned}$$

then from the definition of L_m we have $L_m \equiv F_{r(a_n)-b_n+m} \pmod{a_n}$ for $m=0, 1, 2, \cdots$, and consequently,

$$L_{b_n} \equiv F_{r(a_n)-b_n+b_n} \equiv F_{r(a_n)} \equiv 0 \pmod{a_n}.$$

(Since the a_j are relatively prime in pairs, then the Chinese remainder theorem guarantees the existence of L_0 and L_1 satisfying (2) simultaneously for $n=1, 2, \cdots, 18$.)

I am very grateful to Mr. John Brillhart for his assistance in obtaining an explicit solution to (2). In particular, the smallest positive solution to (2) is given by

$$M = L_0 = 1786772701928802632268715130455793,$$

$$N = L_1 = 1059683225053915111058165141686995.$$

From the way in which M and N were constructed, it follows at once that all the terms of $S(M, N)$ are composite while a routine application of the euclidean algorithm shows that $(M, N) = 1$.

A GEOMETRICAL COINCIDENCE

LEON BANKOFF, Los Angeles

CD is a chord of circle (O) perpendicular to the hypotenuse AB of the inscribed right triangle ABC. The rest of the figure is self-explanatory.

THEOREM. *The diameter of (P) is equal to the sum of the radii of (Q) and (R).*

Proof. (a) Diameter of $(P) = CU + CV = (AC - AU) + (BC - BV) = AC + BC - AB$. (b) Let $AB = 2r$; $AE = 2r_1$; $EB = 2r_2$. Then $OE = r_1 - r_2$. Since $OQ^2 - ON^2 = NQ^2$, we have

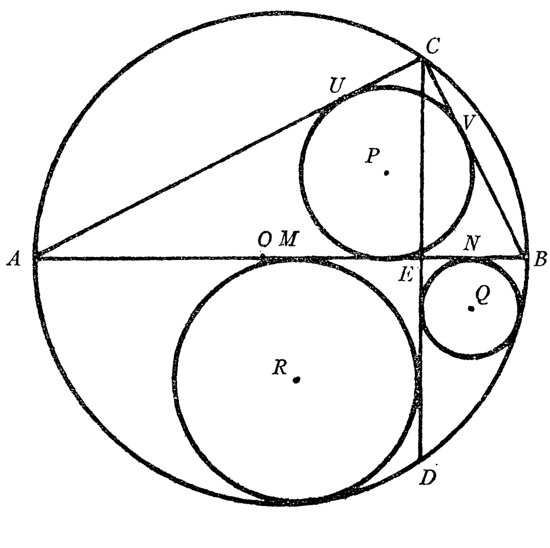
$$(r_1 + r_2 - NQ)^2 - (r_1 - r_2 + NQ)^2 = NQ^2,$$

from which we find

$$NQ = 2\sqrt{(rr_1)} - 2r_1 = AC - AE.$$

Similarly, $RM = CB - EB$. So $RM + NQ = AC + CB - (AE + EB) = AC + CB - AB$.

Note. If CD passes through O, the three inscribed circles are equal.



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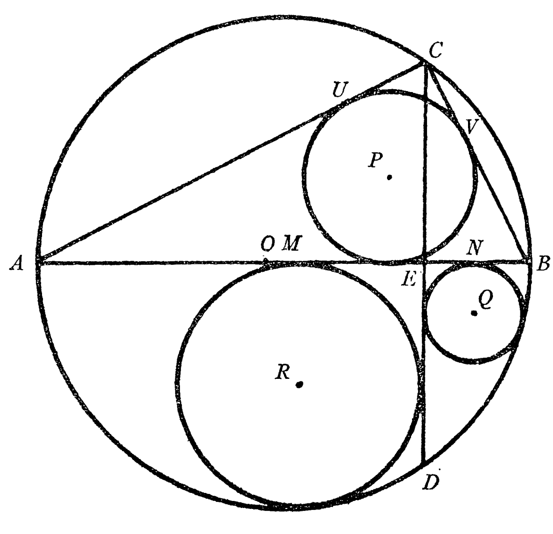

AN INTEGRAL PROPERTY OF CUBICS AND QUADRATICS

WALTER R. TALBOT, Lincoln University of Missouri

If a cubic or quadratic function is integrated between two roots, the result is 2/3 of the product of the difference of the roots and the value of the function midway between those roots. That is, it is to be shown that if $f(x)$ is a cubic or quadratic and a and b are two roots of $f(x)=0$, then

$$\int_a^b f(x)dx = \frac{2}{3}(b-a)f\left(\frac{a+b}{2}\right). \tag{1}$$

Consider the cubic case with roots a, b, c and $p=\Sigma a$, $q=\Sigma ab$, $r=abc$ so that $f(x)\equiv x^3-px^2+qx-r$. It is easily found that

$$(b-a)f\left(\frac{a+b}{2}\right) = \frac{(b-a)^3}{8}(2c-a-b),$$

and

$$\int_a^b f(x)dx = \frac{1}{12}\left[3x^4-4px^3+6qx^2-12rx\right]_a^b = \frac{(b-a)^3}{12}(2c-a-b).$$

The relation (1) is now apparent. If $f(x)$ is quadratic,

$$\int_a^b f(x)dx = \frac{2}{3}(b-a)f\left(\frac{a+b}{2}\right) = \frac{(a-b)^3}{6}. \tag{2}$$

The relation (1) is valid without regard for whether the roots or coefficients of $f(x)=0$ are real or complex or whether in the cubic case the roots are consecutive real roots or not. If the roots are real, the relation (1) is equivalent to each of the next two statements.

If the cubic (or quadratic) $f(x)=0$ has real roots, the net area between the curve $y=f(x)$ and the x-axis, between two roots, equals 2/3 of the product of the distance between those roots and the ordinate midway between those roots.

If the cubic (or quadratic) $f(x)=0$ has real roots, the mean value of $f(x)$ between two roots is 2/3 of the ordinate midway between those roots.

If the roots of the quadratic $f(x)=0$ are real, then by the relation (2) the area between the curve $y=f(x)$ and the x-axis, between the roots, is 1/6 of the cube of the distance between the roots.

A NOTE ON FINITE BOOLEAN RINGS

R. E. SMITHSON, U. S. Naval Ordnance Test Station, China Lake, California

In McCoy [1] it is shown that a finite Boolean ring is isomorphic to a complete direct sum (see [1] for a definition of this term) of 2-element fields and hence, has an identity and is of order 2^n for some n. The proof given in [1] uses rather deep results from ring theory. The purpose of this note is to prove the same result by elementary means.

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A NOTE ON FINITE BOOLEAN RINGS

R. E. SMITHSON, U. S. Naval Ordnance Test Station, China Lake, California

In McCoy [1] it is shown that a finite Boolean ring is isomorphic to a complete direct sum (see [1] for a definition of this term) of 2-element fields and hence, has an identity and is of order 2^n for some n. The proof given in [1] uses rather deep results from ring theory. The purpose of this note is to prove the same result by elementary means.

DEFINITION. *A nontrivial ring R is called a Boolean ring in case each element $x\in R$ satisfies $x^2=x$.*

The proof of Lemma 1 is in [1, pp. 140–141] and is elementary and thus is omitted.

LEMMA 1. *If R is a Boolean ring, then R is commutative and for each $x\in R$ one has $x+x=0$.*

In addition to this we obtain:

LEMMA 2. *Every finite Boolean ring is of order 2^n for some n.*

Proof. Let R be the ring and let p be a prime. By [2, Th. 4.1.1] if p divides the order of R, then, as an additive group, R contains an element of order p. But by Lemma 1 each element of R is of order 2 under addition. Thus the order of R is 2^n for some n.

We now exploit the fact that a finite Boolean ring R can be considered as a vector space over the field of integers modulo 2. In the following R will be a finite Boolean ring of order 2^n.

DEFINITION. *A basis for R is a set $\{x_1, \cdots, x_n\}$ of elements of R such that* (i) *each element of R is the sum of elements of the basis, and* (ii) *each sum of distinct basis elements is nonzero.*

Remark. It can be shown that a basis for R must have n elements and that R must have at least one basis.

Notation. The symbol $\sum_j x_{\alpha_j}$ will be used in the following sense. The index j may range over the empty set in which case $\sum_j x_{\alpha_j}=0$ or, for each j, x_{α_j} is a distinct basis element.

We shall use the following lemma. The proof is straightforward and is omitted.

LEMMA 3. *If $\{x_1, \cdots, x_n\}$ is a basis for R, then*

$$\{x_1, \cdots, x_{i-1}, x_i + \sum_j x_{\alpha_j}, x_{i+1}, \cdots, x_n\}$$

is a basis for R when $\alpha_j\neq i$ for all j.

DEFINITION. *A basis $\{x_1, \cdots, x_n\}$ for R is said to be orthogonal in case $x_i\cdot x_j=0$ if $i\neq j$.*

The desired result rests on the following fundamental lemma.

LEMMA 4. *There exists an orthogonal basis for R.*

Proof. Let $\{x_1, \cdots, x_n\}$ be any basis for R. We shall proceed by induction. First $\{x_1\}$ satisfies $x_i\cdot x_j=0$ for all $i\neq j$ and $i, j\leqq 1$. Thus assume that k is such that $x_i\cdot x_j=0$ for $i\neq j$ and $i, j\leqq k$. We shall construct a new basis such that $x_i'\cdot x_j'=0$ for $i\neq j$ and $i, j\leqq k+1$, where x_i', x_j' are elements of this new basis. If the original basis is incapable of serving as the new one, let j_0 be the least index such that $x_{k+1}x_{j_0}\neq 0$, whence $j_0<k+1$. We shall perform an induction on j_0.

That is, we shall construct a new basis $\{x_1', \cdots, x_n'\}$ such that $x_i' \cdot x_j' = 0$ for $i \neq j$ and $i, j \leqq k$ and such that $x_i' \cdot x_{k+1}' = 0$ for $i \leqq j_0$.

Let $x_{j_0} \cdot x_{k+1} = \sum_i x_{\alpha_i} \neq 0$. There are two cases to be considered.

(1) For all i we have $\alpha_i \neq j_0$. Then set $x_{j_0}' = x_{j_0} + x_{j_0}x_{k+1}$ and $x_i' = x_i$ for $i \neq j_0$. Then $\{x_1', \cdots, x_n'\}$ is the desired basis.

(2) For some i, $\alpha_i = j_0$. In this case set $x_{j_0}' = x_{j_0}x_{k+1}$, $x_{k+1}' = x_{k+1} + x_{j_0}'$ and $x_i' = x_i$ for $i \neq j_0$, $k+1$. Then $\{x_1', \cdots, x_n'\}$ is the desired basis.

The proofs of Case 1 and Case 2 are similar and we give only the proof of Case 2.

In order to see that $\{x_1', \cdots, x_n'\}$ is a basis in case (2) we apply Lemma 3, twice. First we replace x_{j_0} by x_{j_0}'; then replace x_{k+1} by $x_{k+1} + x_{j_0}'$.

Now if $i, j \leqq k$ and $i, j \neq j_0$, then $x_i'x_j' = x_ix_j = 0$. If $j = j_0$ and $i \leqq k$, then $x_i'x_{j_0}' = x_i(x_{j_0}x_{k+1}) = 0$. Thus, $x_i'x_j' = 0$ if $i, j \leqq k$. Finally, $x_i'x_{k+1}' = x_i'x_{k+1} + x_i'x_{j_0}'$. Thus, $x_i'x_{k+1}' = 0$ if $i < j_0$. If $i = j_0$, we have

$$x_i'x_{k+1}' = x_{j_0}'(x_{k+1} + x_{j_0}') = x_{j_0}x_{k+1}x_{k+1} + x_{j_0}'^2 = x_{j_0}x_{k+1} + x_{j_0}' = x_{j_0}' + x_{j_0}' = 0.$$

The induction on j_0 is completed and this induction gives the induction on k and hence the result.

Remark. At this stage we can construct an identity for R. Indeed, if $\{x_1, \cdots, x_n\}$ is an orthogonal basis for R, then $1 = \sum_{i=1}^n x_i$ is an identity for R. However, we can get the stronger result stated in the beginning.

THEOREM. *Every finite Boolean ring R is isomorphic to a complete direct sum of 2-element fields.*

Proof. By Lemmas 2 and 4, R is of order 2^n and has an orthogonal basis. Also, if S is the complete direct sum of n 2-element fields, then S is a Boolean ring of order 2^n. The proof is completed by noting that any one-to-one mapping of an orthogonal basis for R onto an orthogonal basis for any other Boolean ring R' of order 2^n can be extended to an isomorphism of R onto R'.

References

1. N. H. McCoy, Rings and ideals. Carus Math. Monographs, No. 8, 1948.
2. M. Hall, The theory of groups, Macmillan, New York, 1959.

A FOUR CIRCLE INEQUALITY

JAMES M. SLOSS, University of California, Santa Barbara

In this note we are concerned with a geometrical construction and an inequality arising from the construction.

Let C_1 and C_2 be two distinct concentric circles with center O of radii R_1 and R_2 respectively (see figure); assume $R_2 < R_1$. Let P, not O, be any point inside C_2 and let the line L pass through OP. Construct a circle C_4 of radius R_4 where $\overline{OP} + R_4 > R_2 > R_2 - \overline{OP}$ so that C_4 cuts C_2. Let Q be one of the points of

That is, we shall construct a new basis $\{x_1', \cdots, x_n'\}$ such that $x_i' \cdot x_j' = 0$ for $i \neq j$ and $i, j \leqq k$ and such that $x_i' \cdot x_{k+1}' = 0$ for $i \leqq j_0$.

Let $x_{j_0} \cdot x_{k+1} = \sum_i x_{\alpha_i} \neq 0$. There are two cases to be considered.

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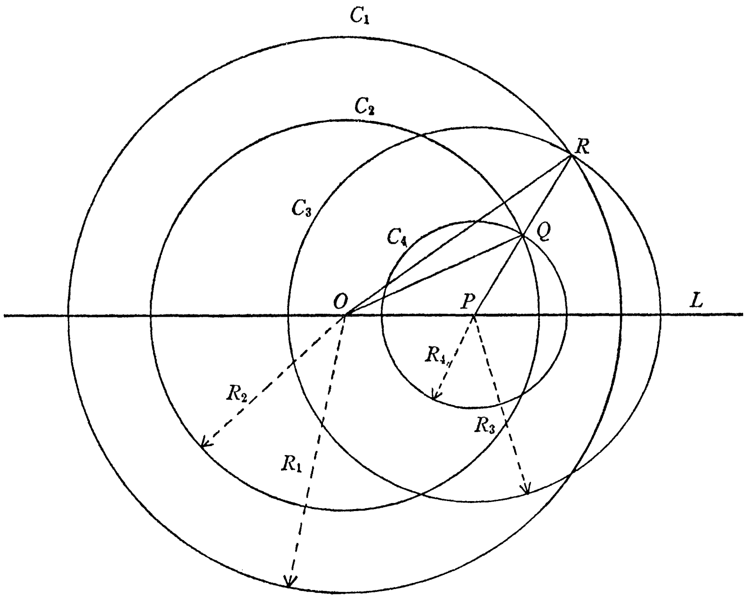

THEOREM.

(i) $\dfrac{\overline{OR}}{\overline{OQ}} < \dfrac{\overline{OP}+\overline{PR}}{\overline{OP}+\overline{PQ}}$ *if and only if* $\overline{PQ}\cdot\overline{PR} < \overline{OP}$;

(ii) $\dfrac{\overline{OR}}{\overline{OQ}} > \dfrac{\overline{OP}+\overline{PR}}{\overline{OP}+\overline{PQ}}$ *if and only if* $\overline{PQ}\cdot\overline{PR} > \overline{OP}$;

(iii) $\dfrac{\overline{OR}}{\overline{OQ}} = \dfrac{\overline{OP}+\overline{PR}}{\overline{OP}+\overline{PQ}}$ *if and only if Q and R are inverse points with respect to the circle centered at P and passing through O.*

Proof. (i) We may as well assume that L lies along the x axis, that O is the origin, and that $\overline{OP}$ has unit length. Then if we let the vector joining P to R be denoted by $\vec{\mathbf{A}}$ and let $\alpha=\overline{PQ}/\overline{PR}$, $0<\alpha<1$, we have that the vector joining P to Q is $\alpha\,\vec{\mathbf{A}}$. Thus the inequality to be proved becomes

$$\frac{|\vec{\mathbf{i}}+\vec{\mathbf{A}}|}{|\vec{\mathbf{i}}+\alpha\vec{\mathbf{A}}|} < \frac{|\vec{\mathbf{i}}|+|\vec{\mathbf{A}}|}{|\vec{\mathbf{i}}|+|\alpha\vec{\mathbf{A}}|}.$$

This is equivalent to showing that

$$\frac{|\vec{i}+\vec{A}|^2}{|\vec{i}+\alpha\vec{A}|^2}<\frac{1+2|\vec{A}|+|\vec{A}|^2}{1+2\alpha|\vec{A}|+\alpha^2|\vec{A}|^2}.$$

If we let $\vec{A}=a_1 i+a_2 j$ the above becomes:

$$\{1+|\vec{A}|^2+2a_1\}\{1+2\alpha|\vec{A}|+\alpha^2|\vec{A}|^2\}$$
$$<\{1+\alpha^2|\vec{A}|^2+2\alpha a_1\}\{1+2|\vec{A}|+|\vec{A}|^2\}$$

which is the same as $(\alpha-1)(|\vec{A}|-a_1)(1-\alpha|\vec{A}|^2)<0$. Since $0<\alpha<1$, $|\vec{A}|=\sqrt{(a_1^2+a_2^2)}>|a_1|\geqq a_1$ and $\overline{PQ}\cdot\overline{PR}<\overline{OP}$ which yields $\alpha|\vec{A}|\ |\vec{A}|<1$, we have shown that the above inequality is indeed valid.

(ii) Follows similarly. Note that equality holds if and only if $|\vec{A}|=a_1$ or $\alpha|\vec{A}|\ |\vec{A}|=1$. But $|\vec{A}|=a_1$ if and only if C_3 is tangent to C_1, which, by construction, is the case if and only if C_4 is tangent to C_2. But $\overline{OP}+R_4>R_2>R_2-\overline{OP}$ so that this is never possible.

In the event that $\alpha|\vec{A}|\ |\vec{A}|=1$, it is clear that Q and R are inverse points with respect to the circle centered at P and passing through O since this circle has radius 1.

We remark finally that the above theorem is valid, care being taken to avoid multiple intersections, if we replace circles by the class S of translationally and shrinkably equivalent star-like closed curves. This follows since the circles were used as a means of construction.

More specifically, let S_ρ be the set of star-like closed curves of the form

$$\sigma(r,z_0)=r\rho(\theta)e^{i\theta}+z_0,\quad 0\leqq\theta\leqq 2\pi,\qquad \rho(0)=\rho(2\pi),$$

where $\rho(\theta)$ is a fixed function, z_0 is any constant complex number, and r any real positive constant. Thus S_ρ is a three parameter family of curves with parameters z_0 and r.

Let $\sigma_{10}=\sigma(R_1, 0)$ and $\sigma_{20}=\sigma(R_2, 0)$ be two elements of S_ρ. Assume $R_2<R_1$. Let P, not O, be any point inside σ_{20} and let the line L pass through O and P. Let $\sigma_{4P}=\sigma(R_4, P)$ be an element of S_ρ such that σ_{4P} cuts σ_{20} in at least one point Q (may cut in many points). Let the line PQ intersect σ_{10} at R. Let $\sigma_{3p}=\sigma(R_3, P)$ be an element of S_ρ that passes through R. Then the theorem holds as stated.

An interesting question arises here: viz., how can we restrict S_ρ so that given any element of S_ρ, $\sigma(r, z_0)$, and any point P within $\sigma(r, z_0)$, all elements of S_ρ of the form $\sigma(s, p)$ intersect $\sigma(r, z_0)$ in at most two points?

It is interesting to note that, corresponding to (i) and (ii), there is a three dimensional analogue. Let S_1 and S_2 be two distinct concentric spheres of radii R_1 and R_2 respectively; assume $R_2<R_1$, with center O. Let P, not O, be any point inside S_2 and let the line L pass through OP. Construct a sphere S_4 of radius R_4 where $\overline{OP}+R_4>R_2>R_2-\overline{OP}$ so that S_4 cuts S_2. Let C_Q be the circle of intersection of S_4 and S_2. Let the cone with apex at P and base C_Q be extended to intersect S_1 in the circle C_R. Construct the sphere S_3 with center P passing through C_R. Let $A(P, C_Q)$ denote the surface area of the cone whose apex is at P and whose base is the circle C_Q. Then

COROLLARY.

$$\frac{A(O, C_R)}{A(O, C_Q)} < \frac{A(P, C_R)}{A(P, C_Q)}$$

if $R_4R_3 \leqq \overline{OP}$.

Proof. Let $2\sigma_R$ be the circumference of C_R and $2\sigma_Q$ be the circumference of C_Q. Then we have:

$$A(O, C_R) = \sigma_R R_1,\ A(O, C_Q) = \sigma_Q R_2; \quad A(P, C_R) = \sigma_R R_3,\ A(P, C_Q) = \sigma_Q R_4,$$

Now from the Theorem (i), (iii) we have:

$$\frac{R_1}{R_2} < \frac{\overline{OP} + R_3}{\overline{OP} + R_4}$$

if $R_4R_3 \leqq \overline{OP}$, i.e.,

$$R_1R_4 + \overline{OP}(R_1 - R_2) \leqq R_2R_3.$$

But $\overline{OP} > 0$ and $R_1 - R_2 > 0$ so that

$$\frac{R_1}{R_2} \leqq \frac{R_3}{R_4} \quad \text{if} \quad R_4R_3 \leqq \overline{OP}.$$

But

$$\frac{A(O, C_R)}{A(O, C_Q)} = \frac{\sigma_R R_1}{\sigma_Q R_2} < \frac{\sigma_R R_3}{\sigma_Q R_4} = \frac{A(P, C_R)}{A(P, C_Q)}.$$

Thus the corollary is proved.

A NOTE ON REPRESENTATIONS OF ABSTRACT GROUPS AS GROUPS OF MOTIONS

ROBERT A. MELTER, Institute for Advanced Study and University of Massachusetts

1. In an article in this MAGAZINE [1], David Ellis proved that any finite or countable group G is isomorphic to a subgroup of the group of motions of a metric space. In this note, we give a shorter proof of Ellis' result without any cardinality restriction. (A modification of Ellis' proof in which the cardinality restriction was eliminated was given in [3].) We also show in an elementary fashion that a finite commutative group is isomorphic to *the* group of *all* motions on a metric space. The latter result also follows from a theorem of R. Frucht [2] to the effect that every finite group is the automorphism group of a graph. Frucht's theorem was extended to the infinite case by G. Sabidussi [4].

intersect S_1 in the circle C_R. Construct the sphere S_3 with center P passing through C_R. Let $A(P, C_Q)$ denote the surface area of the cone whose apex is at P and whose base is the circle C_Q. Then

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$$\frac{A(O, C_R)}{A(O, C_Q)} < \frac{A(P, C_R)}{A(P, C_Q)}$$

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Proof. Let $2\sigma_R$ be the circumference of C_R and $2\sigma_Q$ be the circumference of C_Q. Then we have:

$$A(O, C_R) = \sigma_R R_1, \; A(O, C_Q) = \sigma_Q R_2; \quad A(P, C_R) = \sigma_R R_3, \; A(P, C_Q) = \sigma_Q R_4,$$

Now from the Theorem (i), (iii) we have:

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if $R_4R_3 \leqq \overline{OP}$, i.e.,

$$R_1R_4 + \overline{OP}(R_1 - R_2) \leqq R_2R_3.$$

But $\overline{OP} > 0$ and $R_1 - R_2 > 0$ so that

$$\frac{R_1}{R_2} \leqq \frac{R_3}{R_4} \quad \text{if} \quad R_4R_3 \leqq \overline{OP}.$$

But

$$\frac{A(O, C_R)}{A(O, C_Q)} = \frac{\sigma_R R_1}{\sigma_Q R_2} < \frac{\sigma_R R_3}{\sigma_Q R_4} = \frac{A(P, C_R)}{A(P, C_Q)}.$$

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2. THEOREM 1. *An abstract group G is isomorphic to a subgroup of the group of motions of a metric space.*

Proof. Let G be a group. For $x, y \in G$ set $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. Under this metric every permutation of G is a motion and the group of permutations of G is isomorphic to the group of motions of G. The theorem then follows at once from Cayley's theorem.

3. LEMMA 1. *Every finite cyclic group is isomorphic to the group of all motions of a subset S of the euclidean plane.*

Proof. Let C_n be the cyclic group with n elements. For $n = 1$, let S be a single point. For $n = 2$ let S be a closed line segment. For $n > 2$ let S be a regular n-gon from which has been deleted a point on each edge one-third the distance from the initial vertex of the edge (traversing counterclockwise) to the terminal vertex. The group of motions of S is then the cyclic group generated by a rotation through $2\pi/n$ radians. The reflections which are part of the group of symmetries of the ordinary n-gon have been eliminated by the deletion of the specified points.

THEOREM 2. *Every finite Abelian group is the group of motions of a metric space.*

Proof. Let G be an Abelian group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$. Let $S(p_k^{\alpha_k})$ be a regular polygon of $p_k^{\alpha_k}$ sides and diameter 1, modified as in the proof of Lemma 1, and such that $S(p_k^{\alpha_k}) \cap S p_j^{\alpha_j} = \emptyset$ if $k \neq j$. Let $S = \bigcup_{k=1}^{n} S(p_k^{\alpha_k})$. For x, y in S, let $d(x, y) =$ euclidean distance of x and y if x and y are in the same $S(p_k^{\alpha_k})$, and $d(x, y) = 1$ if x and y are in distinct polygons. It is easy to see that S is a metric space whose group of motions is the direct product of cyclic groups of order $p_k^{\alpha_k}$ $(k = 1, \cdots, n)$ and hence by the Fundamental Theorem of Abelian Groups, this group of motions is isomorphic to G.

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1. David Ellis, On metric representation of groups, this MAGAZINE, 26 (1952) 183–184.
2. R. Frucht, Herstellung von Graphen mit vorgegebener abstrakter Groupe, Compositio Math., 6 (1939) 239–250.
3. Enrique Valle Flores, Observation on a theorem of D. Ellis, Bol. Soc. Mat. Mexicana, 10, Nos. 3–4, 31–32.
4. G. Sabidussi, Groups with given infinite group, Monatsh. Math., 64 (1960) 64–67.

POINT OF INTERSECTION OF TRIANGLE TRANSVERSALS

D. MOODY BAILEY, Princeton, West Virginia

Let two straight lines in the plane of triangle ABC meet sides BC, CA and AB at the points M, N, O and M', N', O' respectively. In the figure, the lines MNO and $M'N'O'$ meet at the point P and the ray AP is constructed to meet side BC at the point D. Segment ON' is drawn to meet AD at the point D'.

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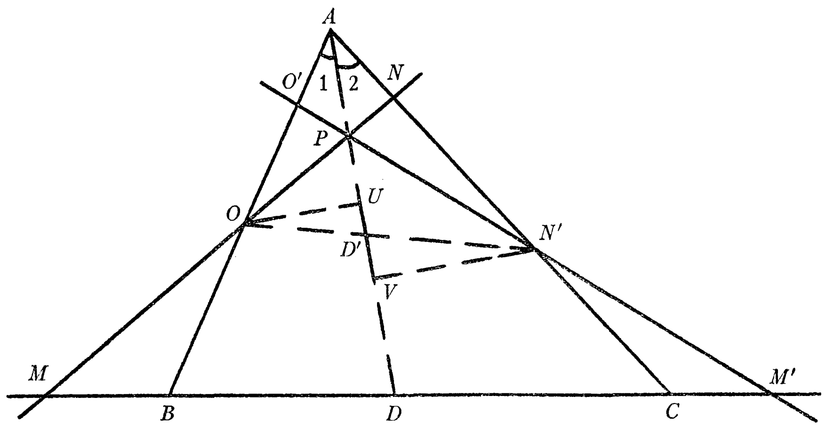
From O and N' perpendiculars to ray AD determine the points U and V respectively. The right triangles OUD' and $N'VD'$ are then similar and $OD'/D'N' = OU/VN'$. A consideration of the right triangles AOU and $AN'V$ yields

$$\frac{OU}{VN'} = \frac{OA}{AN'} \cdot \frac{\sin \angle 1}{\sin \angle 2} = \left(\frac{OO' + O'A}{AN + NN'}\right)\frac{\sin \angle 1}{\sin \angle 2}.$$

Thus

$$\frac{OD'}{D'N'} = \frac{OU}{VN'} = \left(\frac{OO' + O'A}{AN + NN'}\right)\frac{\sin \angle 1}{\sin \angle 2}.$$



In triangle AON', the use of Ceva's theorem yields $(OD'/D'N')\cdot(N'N/NA)\cdot(AO'/O'O) = 1$ or $(OD'/D'N') = (OO'/O'A)\cdot(AN/NN')$. Comparing this value of $OD'/D'N'$ with that given at the end of the preceding paragraph, we see that

$$\left(\frac{OO' + O'A}{AN + NN'}\right)\frac{\sin \angle 1}{\sin \angle 2} = \frac{OO'}{O'A}\cdot\frac{AN}{NN'} \quad \text{or}$$

$$\frac{\sin \angle 1}{\sin \angle 2} = \left(\frac{AN + NN'}{OO' + O'A}\right)\frac{OO'}{O'A}\cdot\frac{AN}{NN'}.$$

It has been shown that $(OD'/D'N') = (OA/AN')\cdot\sin \angle 1/\sin \angle 2$ with respect to triangle AON'. In similar fashion $(BD/DC) = (BA/AC)\cdot\sin \angle 1/\sin \angle 2$ with respect to triangle ABC. Substituting for BA/AC and using the value of $\sin \angle 1/\sin \angle 2$ obtained at the end of the preceding paragraph, we have

$$\frac{BD}{DC} = \left(\frac{BO + OO' + O'A}{AN + NN' + N'C}\right)\left(\frac{AN + NN'}{OO' + O'A}\right)\frac{OO'}{O'A}\cdot\frac{AN}{NN'}.$$

Consider the expression

$$-\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A}.$$

This may be written as

$$
\begin{aligned}
-\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A} &= \frac{BO/OA - BO'/O'A}{CN'/N'A - CN/NA} \\
&= \frac{BO/(OO' + O'A) - (BO + OO')/O'A}{CN'/(N'N + NA) - (CN' + N'N)/NA} \\
&= \frac{OO'}{N'N}\left(\frac{BO + OO' + O'A}{CN' + N'N + NA}\right)\left(\frac{N'N + NA}{OO' + O'A}\right)\frac{NA}{O'A} \\
&= \left(\frac{BO + OO' + O'A}{AN + NN' + N'C}\right)\left(\frac{AN + NN'}{OO' + O'A}\right)\frac{OO'}{O'A}\cdot\frac{AN}{NN'}\,.
\end{aligned}
$$

This final value is obtained by use of the equalities $(N'N+NA) = -(AN+NN')$, $NA = -AN$, $N'N = -NN'$, and $(CN'+N'N+NA) = -(AN+NN'+N'C)$.

It is thus shown that

$$
-\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A}
$$

is equivalent to the value of BD/DC previously determined. Rays BP and CP may be constructed to meet sides CA and AB at points E and F after which corresponding values may be obtained for ratios CE/EA and AF/FB. This result then follows:

THEOREM. *Let two straight lines in the plane of triangle ABC meet sides BC' CA, AB at respective points M, N, O and M', N', O'. Further, let lines MNO and $M'N'O'$ meet at the point P and extend rays AP, BP, CP to meet sides BC, CA, AB at points D, E, F respectively. Then*

$$
\frac{BD}{DC} = -\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A}, \qquad \frac{CE}{EA} = -\frac{CM/MB - CM'/M'B}{AO/OB - AO'/O'B},
$$

$$
\frac{AF}{FB} = -\frac{AN/NC - AN'/N'C}{BM/MC - BM'/M'C}\,.
$$

The results given in this theorem will always be true provided the segments involved are considered as directed quantities. That is, if $M(M')$ lies between B and C, then $BM/MC(BM'/M'C)$ is considered positive. If $M(M')$ lies on BC extended, then $BM/MC(BM'/M'C)$ must be considered negative. Similar comments apply to the other segments involved in the results given.

It is not difficult to show that $(BD/DC)\cdot(CE/EA)\cdot(AF/FB) = 1$ and Ceva's equation is then satisfied for point P. To do this it must be remembered that $(BM/MC)\cdot(CN/NA)\cdot(AO/OB) = -1$ and $(BM'/M'C)\cdot(CN'/N'A)\cdot(AO'/O'B) = -1$ by the theorem of Menelaus. This enables us to replace $(BO/OA)\cdot(CM/MB)$ by $-CN/NA$, etc. The author has found the theorem to be of value in investigating properties of the triangle.

A PROJECTION OF NORM ONE MAY NOT EXIST

DWIGHT B. GOODNER, Florida State University

If H is a subspace in a linear space X, then there exists an idempotent linear operator T with domain X and range H. This operator T is called a projection of X onto H, (cf. [1], p. 241).

Let us assume that X is a normed linear space and let us investigate the norm of the projection T:

$$\|T\| = \sup_{\|x\| \leqq 1} \|Tx\|, \qquad x \in X.$$

We will demonstrate that it is not always possible to find a projection of norm one from a finite dimensional space X onto one of its subspaces the dimension of which is one less than the dimension of X. We will accomplish our goal by constructing an example, the analysis of which will show that a projection of norm one cannot exist.

Let X be the space of all triples of real numbers (x_1, x_2, x_3) where the norm is defined by $\|(x_1, x_2, x_3)\| = \max(|x_1|, |x_2|, |x_3|)$. We choose a subspace H in X.

$$H = \{(x_1, x_2, x_3) \in X: 2x_1 + 2x_2 - 3x_3 = 0\}. \tag{1}$$

Let us assume that there is a projection T of X onto H such that

$$\|T\| = 1. \tag{2}$$

In the following investigation we will make frequent use of the linearity of T and the idempotency of T which we list here for reference purposes:

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty, \qquad x, y \in X, \qquad \alpha, \beta \text{ real}, \tag{3}$$

$$T(Tx) = Tx, \qquad x \in X. \tag{4}$$

Now, if $T(1, 1, 1) = (a_1, a_2, a_3)$, then, because of (2), $\max(|a_1|, |a_2|, |a_3|) = \|T(1, 1, 1)\| \leqq 1$, i.e.,

$$|a_i| \leqq 1, \qquad i = 1, 2, 3. \tag{5}$$

Since $(a_1, a_2, a_3) \in H$, we have in view of (4) that $T(a_1, a_2, a_3) = (a_1, a_2, a_3)$ and consequently in view of (3) that $T(1-a_1, 1-a_2, 1-a_3) = (a_1, a_2, a_3) - (a_1, a_2, a_3) = (0, 0, 0)$. Thus if $T(1, -1, 1) = (b_1, b_2, b_3)$, then $T(1+t(1-a_1), -1+t(1-a_2), 1+t(1-a_3)) = T(1, -1, 1) + tT(1-a_1, 1-a_2, 1-a_3) = (b_1, b_2, b_3)$. Let us determine t so that $(1+t(1-a_1), -1+t(1-a_2), 1+t(1-a_3)) \in H$. Since $(a_1, a_2, a_3) \in H$, we have by (1) that $2a_1 + 2a_2 - 3a_3 = 0$ and consequently $t = 3$. Thus the unique point $h \in H$ for which $Th = (b_1, b_2, b_3)$ is $(4-3a_1, 2-3a_2, 4-3a_3)$. But since $T(b_1, b_2, b_3) = (b_1, b_2, b_3)$ by (4) we have $(b_1, b_2, b_3) = (4-3a_1, 2-3a_2, 4-3a_3)$. Since $T(1, -1, 1) = (b_1, b_2, b_3)$, it follows that $|b_1| = |4-3a_1| \leqq 1$ which in turn implies that $1 \leqq a_1 \leqq 5/3$. Since $|a_i| \leqq 1$ (see (5)), we have $a_1 = 1$. Similarly, we obtain $a_3 = 1$. Thus, $(b_1, b_2, b_3) = (4-3a_1, 2-3a_2, 4-3a_3) = (1, 2-3a_2, 1)$. Since $(b_1, b_2, b_3) \in H$, we obtain from (1) that $a_2 = \frac{1}{2}$. Hence $(a_1, a_2, a_3) = (1, \frac{1}{2}, 1)$ and $(b_1, b_2, b_3) = (1, \frac{1}{2}, 1)$. Therefore $T(1, 1, 1) = T(1, -1, 1) = (1, \frac{1}{2}, 1)$.

In view of the linearity of T (see (3)), $T(0, \frac{1}{2}, 0)=T(1, 1, 1)-T(1, \frac{1}{2}, 1)=(0, 0, 0)$. (Note that $(1, \frac{1}{2}, 1)\in H$ and hence $T(1, \frac{1}{2}, 1)=(1, \frac{1}{2}, 1)$). Now, if $T(-1, 1, 1)=(c_1, c_2, c_3)$, then, because $T(c_1, c_2, c_3)=(c_1, c_2, c_3)$ and $T(-1+0, 1+k\frac{1}{2}, 1+0)=T(-1, 1, 1)+kT(0, \frac{1}{2}, 0)=T(-1, 1, 1)$, we can use an argument that is analogous to the one pertaining to the evaluation of the b_i's to show that $(c_1, c_2, c_3)=(-1, \frac{5}{2}, 1)$. Thus $T(-1, 1, 1)=(-1, \frac{5}{2}, 1)$, i.e., $\|T(-1, 1, 1)\|=\frac{5}{2}$. Since $\|T\|=1$ by hypothesis, we have $\frac{5}{2}\leqq 1$ which is impossible. Hence it follows that there is no projection T of norm one from this particular space X to this particular subspace H.

Reference

1. A. E. Taylor, Functional analysis, Wiley, New York, 1958.

ON ROUND PEGS IN SQUARE HOLES AND SQUARE PEGS IN ROUND HOLES

DAVID SINGMASTER, University of California, Berkeley

Some time ago, the following problem occurred to me: which fits better, a round peg in a square hole or a square peg in a round hole? This can easily be solved once one arrives at the following mathematical formulation of the problem. Which is larger: the ratio of the area of a circle to the area of the circumscribed square or the ratio of the area of a square to the area of the circumscribed circle? One easily finds that the first ratio is $\pi/4$ and that the second is $2/\pi$. Since the first is larger, we may conclude that a round peg fits better in a square hole than a square peg fits in a round hole.

More recently, it occurred to me that the above question could be easily generalized to n dimensions. The remainder of this paper will be devoted to the following

THEOREM. *The n-ball fits better in the n-cube than the n-cube fits in the n-ball if and only if* $n\leqq 8$.

First, we take the following formula for the n-volume of the n-ball of radius r [1, p. 136]

$$V=\frac{\pi^{n/2}r^n}{\Gamma(n/2+1)},$$

where $\Gamma(x)$ is the well-known gamma function. (It is noteworthy that the volume of the unit n-ball decreases to zero with increasing n.)

Since we are interested only in ratios, we may, without loss of generality, assume that we have the unit n-ball in both ratios. Then the edge of the circumscribed cube is 2. Since the diagonal of an n-cube is $\sqrt{n}$ times its edge, we see that the edge of the n-cube which is inscribed in the unit n-ball is $2/\sqrt{n}$. Letting $V(n)$, $V_c(n)$, and $V_i(n)$ represent the n-volumes of the unit n-ball, its circum-

In view of the linearity of T (see (3)), $T(0, \frac{1}{2}, 0) = T(1, 1, 1) - T(1, \frac{1}{2}, 1) = (0, 0, 0)$. (Note that $(1, \frac{1}{2}, 1) \in H$ and hence $T(1, \frac{1}{2}, 1) = (1, \frac{1}{2}, 1)$). Now, if $T(-1, 1, 1) = (c_1, c_2, c_3)$, then, because $T(c_1, c_2, c_3) = (c_1, c_2, c_3)$ and $T(-1+0, 1+k\frac{1}{2}, 1+0) = T(-1, 1, 1) + kT(0, \frac{1}{2}, 0) = T(-1, 1, 1)$, we can use an argument that is analogous to the one pertaining to the evaluation of the b_i's to show that $(c_1, c_2, c_3) = (-1, \frac{5}{2}, 1)$. Thus $T(-1, 1, 1) = (-1, \frac{5}{2}, 1)$, i.e., $\|T(-1, 1, 1)\| = \frac{5}{2}$. Since $\|T\| = 1$ by hypothesis, we have $\frac{5}{2} \leqq 1$ which is impossible. Hence it follows that there is no projection T of norm one from this particular space X to this particular subspace H.

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Since we are interested only in ratios, we may, without loss of generality, assume that we have the unit n-ball in both ratios. Then the edge of the circumscribed cube is 2. Since the diagonal of an n-cube is $\sqrt{n}$ times its edge, we see that the edge of the n-cube which is inscribed in the unit n-ball is $2/\sqrt{n}$. Letting $V(n)$, $V_c(n)$, and $V_i(n)$ represent the n-volumes of the unit n-ball, its circumscribed n-cube, and its inscribed n-cube, respectively, we have:

$$V(n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}, \qquad V_c(n) = 2^n, \qquad V_i(n) = \frac{2^n}{n^{n/2}}.$$

Now, the ratios under consideration are:

$$R_1(n) = \frac{V(n)}{V_c(n)} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)\cdot 2^n}, \qquad R_2(n) = \frac{V_i(n)}{V(n)} = \frac{2^n\Gamma\left(\frac{n+2}{2}\right)}{n^{n/2}\pi^{n/2}}.$$

The ratio $R_1(n)$ measures how well an n-ball fits in an n-cube and the ratio $R_2(n)$ measures how well an n-cube fits in an n-ball. Our theorem can now be stated as: $R_1(n) \geqq R_2(n)$ if and only if $n \leqq 8$. We shall prove somewhat more.

THEOREM. *$R_1(n)/R_2(n) \to 0$ as $n \to \infty$.*

Proof. From the definitions, we have:

$$\frac{R_1(n)}{R_2(n)} = \frac{\pi^n n^{n/2}}{2^{2n}\left[\Gamma\left(\frac{n+2}{2}\right)\right]^2}.$$

We apply Stirling's approximation: $\Gamma(z) \sim z^{z-1/2}e^{-z}\sqrt{(2\pi)}$, thus obtaining

$$\left[\Gamma\left(\frac{n+2}{2}\right)\right]^2 \sim \left(\frac{n+2}{2}\right)^{n+1} e^{-n-2}2\pi.$$

Hence, we have:

$$\frac{R_1(n)}{R_2(n)} \sim \frac{\pi^n n^{n/2}}{2^{2n}\left(\frac{n+2}{2}\right)^{n+1} e^{-n-2}2\pi} = \frac{e^2}{\pi(n+2)}\left[\frac{\pi e\sqrt{n}}{2(n+2)}\right]^n < \frac{e^2}{\pi n}\left[\frac{\pi e}{2}\cdot\frac{1}{\sqrt{n}}\right]^n.$$

This last quantity is easily seen to approach zero as n increases, hence the theorem is proved.

COROLLARY. *$R_1(n) < R_2(n)$ for all large enough n.*

One may readily compute that the asymptotic approximation for $R_1(n)/R_2(n)$ has the value 1.06 . . . for $n=8$ and the value .84 . . . for $n=9$. Since Stirling's approximation has a relative error less than $1/12z$, we can say that the relative error in the asymptotic expression for $R_1(n)/R_2(n)$ is less than $2\times 1/12\cdot 5$ or 3.3% for $n \geqq 8$. Hence we can be confident in stating that $R_1(n) < R_2(n)$ holds if $n \geqq 9$, since the asymptotic approximation decreases with n, for $n \geqq 5$. That the asymptotic approximation decreases with n is clear when $n \geqq 14$ since $\sqrt{n}/(n+2)$ decreases with n for $n \geqq 2$ and

$$\frac{\pi e\sqrt{n}}{2(n+2)} < 1$$

for $n \geqq 14$. Further, one can compute that the asymptotic approximation decreases in the range $5 \leqq n \leqq 14$. Hence the theorem first stated has been half proven.

In order to check the theorem for small values of n, I programmed the IBM 7090 computer at Berkeley to compute $V(n)$, $R_1(n)$ and $R_2(n)$ for $1 \leqq n \leqq 100$. The results, which are partially reproduced below, show that $R_1(n) \geqq R_2(n)$ holds if $n \leqq 8$, with equality only for $n=1$. The numerical results for small n, together with the asymptotic results for large n, show that $R_1(n) \geqq R_2(n)$ if and only if $n \leqq 8$, as originally claimed.

n	$V(n)$	$R_1(n)$	$R_2(n)$
1	2.0	1.0	1.0
2	3.14159	.78540	.63662
3	4.18879	.52360	.36755
4	4.93479	.30842	.20264
5	5.26378	.16449	.10875
6	5.16770	.080745	.057336
7	4.72475	.036912	.029853
8	4.05870	.015854	.015399
9	3.29850	.0064423	.0078861
10	2.55015	.0024904	.0040154
11	1.88410	.00091997	.0020350
12	1.33526	.00032599	.0010273
13	.91062	.00011116	.00051691
14	.59926	.000036576	.00025936
15	.38144	.000011641	.00012982
16	.23533	.0000035908	.000064840
17	.14098	.0000010756	.000032325
18	.082145	.00000031336	.000016088
19	.046621	.000000088923	.0000079952
20	.025807	.000000024611	.0000039680
30	.000021915	2.0410×10^{-14}	3.4146×10^{-9}
40	3.6047×10^{-9}	3.2784×10^{-21}	2.7741×10^{-12}
50	1.7302×10^{-13}	1.5367×10^{-28}	2.1835×10^{-15}
60	3.0962×10^{-18}	2.6856×10^{-36}	1.6844×10^{-18}

In closing, we remark that one can also show that $R_2(n)$ and $V(n)/R_2(n)$ each approach zero with $V(n) > R_2(n)$ if and only if $n \leqq 61$.

Reference

1. D. M. Y. Sommerville, An introduction to the geometry of N dimensions, Dover, New York, 1958.

REFLECTIONS ON PURE GEOMETRY

NATHAN ALTSHILLER COURT, University of Oklahoma

1. Analytical solutions of problems in euclidean geometry are often cumbersome, according to Seymour Schuster (this MAGAZINE, 36 (1963) 81–83). More satisfactory solutions may be obtained by using methods furnished by synthetic for $n \geqq 14$. Further, one can compute that the asymptotic approximation decreases in the range $5 \leqq n \leqq 14$. Hence the theorem first stated has been half proven.

In order to check the theorem for small values of n, I programmed the IBM 7090 computer at Berkeley to compute $V(n)$, $R_1(n)$ and $R_2(n)$ for $1 \leqq n \leqq 100$. The results, which are partially reproduced below, show that $R_1(n) \geqq R_2(n)$ holds if $n \leqq 8$, with equality only for $n=1$. The numerical results for small n, together with the asymptotic results for large n, show that $R_1(n) \geqq R_2(n)$ if and only if $n \leqq 8$, as originally claimed.

n	$V(n)$	$R_1(n)$	$R_2(n)$
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1. Analytical solutions of problems in euclidean geometry are often cumbersome, according to Seymour Schuster (this MAGAZINE, 36 (1963) 81–83). More satisfactory solutions may be obtained by using methods furnished by synthetic

projective geometry, and this procedure would, in addition, provide a deeper insight into the problem thus treated.

To prove his point Schuster discusses two problems which were solved analytically in the Putnam Competitions. Here is the first problem:

Given an acute-angled triangle ABC and one altitude AH, select any point D on AH then draw BD and extend it until it intersects AC in E. Draw CD and extend until it intersects AB in F. Prove that angle AHE = angle AHF.

Using the harmonic properties of the complete quadrangle $ABCD$, Schuster disposes of the proof of the proposition in three lines of print.

2. Another synthetic proof of the proposition may be obtained as follows: The point D has EFH for its cevian triangle with respect to the triangle ABC; hence the two corresponding sides EF, BC of the two triangles meet in the harmonic conjugate H' of H for the two points B, C (**1**; pp. 244, 245). We thus have two pairs of harmonic lines: HE, HF; HD, HH' and the latter pair is rectangular; hence the proposition [**1**; p. 170, art. 355].

3. The problem considered belongs to the domain of elementary euclidean geometry and may be solved by the methods and learning which Euclid provides in his Elements.

Let the lines HE, HF, AC, AB cut the parallel through D to the side BC in the points P, Q, R, S.

The segments determined on the two parallels BC, RS by the triads of lines passing through each of the three points E, F, A, give rise to the three proportions: $DP:DR=BH:BC$, $DS:DQ=BC:CH$, and $DR:DS=HC:HB$. Multiplying the three proportions term by term and simplifying we obtain $DP:DQ=1$.

Thus in the triangle HPQ the altitude HD bisects the base PQ; hence HPQ is isosceles, and therefore HDA bisects the angle $PHQ \equiv EHF$. The proposition is proved.

Other elementary proofs may be obtained by considering the two perpendiculars from E, F to the base BC of ABC, [**2**].

4. Even hard to please readers could not be expected to object to any one of the three proofs given above on the ground of it being "cumbersome."

Schuster deplores that the sophisticated participants in the Putnam Competitions ignore the synthetic method of solving geometric problems. What is worse, those collegians, presumably the most gifted of the crop, mathematically speaking, are not expected by their elders to know anything about those simple methods.

It seems unlikely to me that this problem would have been included had those who select problems been aware of the possibility of such simple solutions, or of the fact that the problem is proposed as an exercise in a book on pure geometry [**1**, p. 32, ex. 58].

We are prone to forget what analysis owes to geometry and are content to render this neglected discipline just lip service by speaking of "spaces," "linear problems," etc.

The beauty and fascination of pure geometry will again come into its own.

One feels sorry for a generation that will have to master a relatively extensive analytical apparatus before it can become acquainted with the fruits of that geometry, and then know only the dry bones of it. Those who will not have acquired in time that analytical skill will have to get along even without the bones. For what sins is that punishment to be meted out to those students?

References

1. N. A. Court, College geometry, 2nd ed. Barnes and Noble, New York, 1952.
2. Bulletin des sciences physiques et mathématiques élémentaires, 7(1901–1902) 137, 151, 154.

A NOTE ON THE EQUATION $n^2+n+1=p^r$

JAMES P. BURLING AND VICTOR H. KEISER, University of Colorado

In a recent article [1] of this journal the equation $n^2+n-1=N$ for n, N natural numbers was discussed. The equation $n^2+n+1=p^r$ for n, r natural numbers and p a prime arises out of the study of certain finite projective planes. (In particular those with solvable group of collineations of odd order which are primitive considered as permutation groups on the points.) This note is confined to remarks concerning solutions from a number theoretical point of view.

THEOREM 1. *r is odd.*

Proof. From $n^2+n+1=p^r$ we get $4n^2+4n+1=4p^r-3$. Suppose r is even. Then $4p^r$ is a square. But three less than a square number is either 1 or a nonsquare. Since $(2n+1)^2\neq 1$, we conclude that $4p^r-3$ is not a square for even r, and so $(2n+1)^2\neq 4p^r-3$.

THEOREM 2. *p is odd.*

Proof. Clearly n^2+n+1 is always odd.

THEOREM 3. $n\equiv 0$ or 2 (mod 3).

Proof. Suppose $n\equiv 1$ (mod 3). Then $n=3k+1$ and $n^2+n+1=3(3k^2+3k+1)$. Since 3 does not divide $3k^2+3k+1$, p^r is divisible by more than one prime.

THEOREM 4. $p\equiv 1$ (mod 3).

Proof. Since $n\equiv 0$ or 2 (mod 3), $n^2+n+1\equiv 1$ (mod 3) and $p^r\equiv 1$ (mod 3). $p=3$ implies $p^r\equiv 0$ (mod 3); $p\equiv -1$ (mod 3) implies that $p^r\equiv 1$ (mod 3) if and only if r is even. This leaves $p\equiv 1$ (mod 3).

THEOREM 5. $p\equiv 1$ (mod 6).

Proof. Apply Theorem 2 and Theorem 4.

THEOREM 6. $p\not\equiv -1$ (mod 5).

One feels sorry for a generation that will have to master a relatively extensive analytical apparatus before it can become acquainted with the fruits of that geometry, and then know only the dry bones of it. Those who will not have acquired in time that analytical skill will have to get along even without the bones. For what sins is that punishment to be meted out to those students?

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A NOTE ON THE EQUATION $n^2+n+1=p^r$

JAMES P. BURLING AND VICTOR H. KEISER, University of Colorado

In a recent article [1] of this journal the equation $n^2+n-1=N$ for n, N natural numbers was discussed. The equation $n^2+n+1=p^r$ for n, r natural numbers and p a prime arises out of the study of certain finite projective planes. (In particular those with solvable group of collineations of odd order which are primitive considered as permutation groups on the points.) This note is confined to remarks concerning solutions from a number theoretical point of view.

THEOREM 1. *r is odd.*

Proof. From $n^2+n+1=p^r$ we get $4n^2+4n+1=4p^r-3$. Suppose r is even. Then $4p^r$ is a square. But three less than a square number is either 1 or a nonsquare. Since $(2n+1)^2\neq 1$, we conclude that $4p^r-3$ is not a square for even r, and so $(2n+1)^2\neq 4p^r-3$.

THEOREM 2. *p is odd.*

Proof. Clearly n^2+n+1 is always odd.

THEOREM 3. $n\equiv 0$ or 2 (mod 3).

Proof. Suppose $n\equiv 1$ (mod 3). Then $n=3k+1$ and $n^2+n+1=3(3k^2+3k+1)$. Since 3 does not divide $3k^2+3k+1$, p^r is divisible by more than one prime.

THEOREM 4. $p\equiv 1$ (mod 3).

Proof. Since $n\equiv 0$ or 2 (mod 3), $n^2+n+1\equiv 1$ (mod 3) and $p^r\equiv 1$ (mod 3). $p=3$ implies $p^r\equiv 0$ (mod 3); $p\equiv -1$ (mod 3) implies that $p^r\equiv 1$ (mod 3) if and only if r is even. This leaves $p\equiv 1$ (mod 3).

THEOREM 5. $p\equiv 1$ (mod 6).

Proof. Apply Theorem 2 and Theorem 4.

THEOREM 6. $p\not\equiv -1$ (mod 5).

Proof. Suppose $p \equiv -1 \pmod 5$. Then $p^r \equiv -1 \pmod 5$ since r is odd. But $n^2+n+1 \not\equiv -1 \pmod 5$ for any n.

THEOREM 7. $p \equiv 1, 7, 13 \pmod{30}$.

Proof. Apply Theorem 4, Theorem 5, and Theorem 6.

REMARKS. Similar arguments yield further restrictions on p; e.g., $p \not\equiv -1$ (mod 13), but these seem of little interest.

Using the IBM computer at the University of Colorado, we have found that for $r=1$ among the first 12,000 primes there are 83 solutions, the largest being $n=357$. Then $n^2+n+1=127{,}807$ which is the 11,971st prime.

On the other hand, for $r>1$ there is only one solution for $n<180{,}000$: $n=18$, $p=7$ and $r=3$.

Reference

1. C. W. Trigg, The nature of $N=n(n+1)-1$, this MAGAZINE, 36 (1963) 120.

TANGENT OUTSIDE THE CURVE

WILLIAM R. RANSOM, Tufts University

There are various methods of establishing equations of tangents without the use of limits: the author gave one such method on page 159 of the *American Mathematical Monthly* for Jan.-Feb. 1956. Here is another method: y_c and y_t are ordinates of curve and tangent for the same abscissa.

For the circle, $x^2+y^2=r^2$, the tangent at (p, q) is $y_t=q-(x-p)p/q$, for

$$y_t - y_c = (q^2 - px + p^2)/q - \sqrt{(r^2 - x^2)} = [r^2 - px - q\sqrt{(r^2 - x^2)}]/q$$
$$= [\sqrt{(r^4 + p^2x^2 - 2r^2px)} - \sqrt{(r^4 + p^2x^2 - r^2(p^2 + x^2))}]/q.$$

This is zero at the contact point where $x=p$; but when $x\neq p$, $(x-p)^2>0$, $x^2+p^2>2px$, the first square root above exceeds the second, and y_t-y_c is positive on both sides of $x=p$.

The ellipse and hyperbola $x^2/a^2 \pm y^2/b^2=1$ can be handled in the same way, for the y_t-y_c equals (b^2/a^2) times the same bracket that appeared for the circle.

For the parabola $y=kx^2$, $y_t-y_c=kp^2+2kp(x-p)-kx^2$ which reduces to $-k(x-p)^2$, always negative except at the contact where $x=p$.

This method adapts to the case of $y=Kx^n$, where n is positive and rational. The algebra of the general case is tedious but is adequately illustrated for a 5/3 power, $y^3=k^3x^5$. To avoid fractional exponents, use a^3 for p and z^3 for x: this makes the curve's equation $y=kz^5$, (p, q) becomes (a^3, ka^5), and for the slope of the tangent $5ka^2/3$.

$$y_t - y_c = ka^5 + (5/3)ka^2(z^3 - a^3) - kz^5 = [3(a^5 - z^5) + 5a^2(z^3 - a^3)]k/3$$
$$= (a - z)[3(a^4 + a^3z + a^2z^2 + az^3 + z^4) - 5a^2(a^2 + az + z^2)]k/3$$

Proof. Suppose $p\equiv -1 \pmod 5$. Then $p^r\equiv -1 \pmod 5$ since r is odd. But $n^2+n+1\not\equiv -1 \pmod 5$ for any n.

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This is zero at the contact point where $x=p$; but when $x\neq p$, $(x-p)^2>0$, $x^2+p^2>2px$, the first square root above exceeds the second, and y_t-y_c is positive on both sides of $x=p$.

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$$\begin{aligned} y_t - y_c &= ka^5 + (5/3)ka^2(z^3 - a^3) - kz^5 = [3(a^5 - z^5) + 5a^2(z^3 - a^3)]k/3 \\ &= (a - z)[3(a^4 + a^3z + a^2z^2 + az^3 + z^4) - 5a^2(a^2 + az + z^2)]k/3 \end{aligned}$$

To show that the bracket above is divisible by $(z-a)$ with a positive quotient, write the positive terms in three rows and the negative terms in five columns, thus:

$$\begin{array}{l} a^4 - a^4 + a^3z - a^4 \ + a^2z^2 - a^4 \ + az^3 - a^4 \ \ + z^4 - a^4 \\ a^4 - a^2z + a^3z - a^3z \ + a^2z^2 - a^3z + az^3 - a^3z \ + z^4 - a^3z \\ a^4 - a^2z^2 + a^3z - a^2z^2 + a^2z^2 - a^2z^2 + az^3 - a^2z^2 + z^4 - a^2z^2. \end{array}$$

The first three columns reduce to zero, and the last two to

$$(z-a)\begin{bmatrix} a(z^2+az+a^2) & + z^3+a^2z+az^2+a^3 \\ + az(z+a) & + z(z^2+az+a^2) \\ +az^2 & + z^2(z+a) \end{bmatrix}.$$

Hence $y_t - y_c = -(a-z)^2$ (the sum of 15 positive terms) $(k/3)$. If n is less than 1 the same method applies to $y_t - y_c$.

For the hyperbola $xy = k^2$, $y_t - y_c$ reduces to $-k^2(x-p)^2/p^2x$ which is negative in the first quadrant and positive in the third.

For $y = Kx^n$ when n is rational and negative, in the illustrative case, $x^5y^3 = k^3$, taking (p, q) as (a^3, q), x as z^3 and the slope of the tangent as $-5k/3a^8$, gives $y_t - y_c$ as $k/(3a^8z^5)[3a^3(z^5-a^5) - 5z^5(z^3-a^3)]$, and the rest proceeds as in the case where n is positive.

A NOTE ON THE FAIR DIVISION PROBLEM

A. M. FINK, University of Nebraska

1. This note gives a possibly new solution to the fair division problem of Steinhaus [1]. See [2] for a more recent discussion of the problem with an extensive bibliography.

Problem 1. Divide a cake among n people so that each is assured a portion of the cake that he judges to be at least $1/n$ of the cake.

It is convenient to introduce some notation. Let $m_j(a)$ be the measure assigned by player j to the piece a. We assume no relationships between the various measures. Thus the object of the scheme will be to assure that if a_j is the piece that player j will get, then $m_j(a_j) \geqq 1/n$. Our solution will consist of a finite $(n(n-1)/2)$ sequence of division problems each of which involves only two players. We shall assume that each measure is additive, nonatomic, and the total measure of the cake is one.

2. The two player schemes that will be used are solutions to the auxiliary.

Problem 2_k. Divide a cake between two players so that player c is assured a portion of the cake that he judges to be at least $1/k$ of the cake and player d is assured a portion of the cake that he judges to be at least $(k-1)/k$ of the cake.

To show that the bracket above is divisible by $(z-a)$ with a positive quotient, write the positive terms in three rows and the negative terms in five columns, thus:

$$a^4 - a^4 + a^3z - a^4 + a^2z^2 - a^4 + az^3 - a^4 + z^4 - a^4$$
$$a^4 - a^2z + a^3z - a^3z + a^2z^2 - a^3z + az^3 - a^3z + z^4 - a^3z$$
$$a^4 - a^2z^2 + a^3z - a^2z^2 + a^2z^2 - a^2z^2 + az^3 - a^2z^2 + z^4 - a^2z^2.$$

The first three columns reduce to zero, and the last two to

$$(z-a)\begin{bmatrix} a(z^2+az+a^2) & + z^3 + a^2z + az^2 + a^3 \\ + az(z+a) & + z(z^2+az+a^2) \\ + az^2 & + z^2(z+a) \end{bmatrix}.$$

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(k is a positive integer.) In our previous notation we want to divide the cake into two pieces a_c and a_d such that

$$m_c(a_c) \geqq \frac{1}{k} \quad \text{and} \quad m_d(a_d) \geqq \frac{k-1}{k} \cdot$$

The scheme consists of player d dividing the cake into k pieces and player c selecting one piece with player d receiving the remaining portions. The scheme is fair since player d can divide the cake into k pieces $a_1, a_2, \cdots, a_k$ with $m_d(a_i)=1/k$, $i=1, \cdots, k$. Player c should select a piece a_j such that $m_c(a_j) \geqq 1/k$. Such a piece exists since $\sum_{i=1}^{k} m_c(a_i)=1$.

3. We now give a solution to problem 1. The scheme can best be described as an n-stage scheme. At stage 1 the cake is given to player 1. At stage 2, player 2 and player 1 play the auxiliary scheme 2_2. That is, the cake is divided into pieces a_1, a_2 such that $m_1(a_1) \geqq \frac{1}{2}$ and $m_2(a_2) \geqq \frac{1}{2}$. Then at stage 3, player 3 plays, as player c, the auxiliary scheme 2_3 with player 1 and then with player 2. That is, player 3 vies for a piece b_1 of a_1 such that $m_3(b_1) \geqq \frac{1}{3} m_3(a_1)$ and a piece b_2 of a_2 such that $m_3(b_2) \geqq \frac{1}{3} m_3(a_2)$. Meanwhile player 1, as player d, retains a piece c_1 of a_1 such that $m_1(c_1) \geqq \frac{2}{3} m_1(a_1)$ and player 2, as player d, retains a piece c_2 of a_2 such that $m_2(c_2) \geqq \frac{2}{3} m_2(a_2)$. In general, at the kth stage, the kth player participates as player c with each of the players $1, 2, \cdots, k-1$ in the auxiliary scheme 2_k. The scheme ends at the nth stage.

We now show that this scheme is fair to player j. Since player j does not participate until stage j it is sufficient to show that the kth stage where $j \leqq k \leqq n$ is fair to player j. We show that at stage k player j can have a piece such that $m_j(a) \geqq 1/k$. The proof is by induction.

The jth stage is fair to player j if $j=1$. This is trivial, so assume $j>1$. Let $a_1, \cdots, a_{j-1}$ be the pieces owned by players $1, \cdots, j-1$ at the end of stage $j-1$. The auxiliary scheme assures player j a portion b_i, $i=1, \cdots, j-1$, of each of these such that $m_j(b_i) \geqq 1/j\, m_j(a_i)$, $i=1, \cdots, j-1$. Since

$$\sum_{i=1}^{j-1} m_j(a_i) = 1$$

it follows that

$$\sum_{i=1}^{j-1} m_j(b_i) \geqq \frac{1}{j} \cdot$$

The $(k-1)$th stage fair to player j implies the kth stage is fair. If the player j has a piece a at the end of the $(k-1)$th stage then the induction hypothesis is $m_j(a) \geqq 1/(k-1)$. The auxiliary scheme now assures player j (as player d) a piece b such that $m_j(b) \geqq (k-1)/k\, m_j(a) \geqq 1/k$. The proof is now complete. At stage n, $m_j(a) \geqq 1/n$ for each n.

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A NOTE ON THE OPERATOR D

JOHN G. CHRISTIANO AND R. J. CORMIER, Northern Illinois University

Modern texts in differential equations seem to omit certain differential operator relations which are rather interesting and which give the reader an opportunity to exhibit elegant induction proofs and a reasonable degree of sophistication in algebraic manipulation.

In this note, it is assumed that $\binom{n}{r}$ and ${}_nP_r$ are zero for $r>n$. D is the usual differential operator d/dx. All functions are assumed to have continuous derivatives where needed. Furthermore, V denotes a function of x, and L a polynomial $\sum_{i=0}^{n} a_iD^i$ with constant coefficients. L' and $L^{(i)}$ denote the first and ith derivatives of L with respect to D. L^{-1} is the inverse of the operator L, i.e., $LL^{-1}(V) = V$. The proof of the first lemma is left for the reader.

LEMMA 1. $D^n(xV)=xD^n(V)+nD^{n-1}(V)$.

LEMMA 2. $L(xV)=xL(V)+L'(V)$.

Proof.

$$\begin{aligned} L(xV) &= \sum_{i=0}^{n} a_iD^i(xV) = \sum_{i=0}^{n} a_ixD^i(V) + \sum_{i=0}^{n} ia_iD^{i-1}(V) \\ &= xL(V) + L'(V). \end{aligned}$$

LEMMA 3.

$$D^m(x^nV) = \sum_{i=0}^{m} \binom{m}{i} {}_nP_{m-i}x^{n-m+i}D^i(V).$$

Proof. For $m=1$, the lemma is obvious. Assuming the theorem true for $m=k$ and differentiating, we obtain:

$$\begin{aligned} D^{k+1}(x^nV) &= D\sum_{i=0}^{k} \binom{k}{i} {}_nP_{k-i}x^{n-k+i}D^i(V) \\ &= \sum_{i=0}^{k}\left[\binom{k}{i} {}_nP_{k-i+1}x^{n-k+i-1}D^i(V) + \binom{k}{i} {}_nP_{k-i}x^{n-k+i}D^{i+1}(V)\right]. \end{aligned}$$

Now, making a change of index in the second summation we obtain:

$$\begin{aligned} D^{k+1}(x^nV) = {} & \sum_{i=0}^{k} \binom{k}{i} {}_nP_{k-i+1}x^{n-k+i-1}D^i(V) + \sum_{i=1}^{k} \binom{k}{i-1} {}_nP_{k-i+1}x^{n-k+i-1}D^i(V) \\ & + \binom{k+1}{k+1} {}_nP_0x^nD^{k+1}(V). \end{aligned}$$

Combining these sums with the use of the relation $\binom{n}{i}+\binom{n}{i-1}=\binom{n+1}{i}$ we conclude that

$$D^{k+1}(x^nV) = \sum_{i=0}^{k+1} \binom{k+1}{i} {}_nP_{k+1-i}x^{n-k+i-1}D^i(V)$$

and, hence, the lemma is proved.

THEOREM 1.

$$L(x^nV) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}L^i(V).$$

Proof. The theorem is valid for $n=1$. Assuming the validity for $n=k$, we get

$$\begin{aligned} L(x^{k+1}V) &= xL(x^kV) + L'(x^kV) \\ &= \sum_{i=0}^{k} \binom{k}{i} x^{k+1-i}L^i(V) + \sum_{i=0}^{k} \binom{k}{i} x^{k-i}L^{i+1}(V). \end{aligned}$$

Noting that $\binom{k}{k+1}x^0L^{k+1}(V)=0$, we obtain the following by isolating the $i=0$ term from the first summation and shifting the index in the second summation:

$$\begin{aligned} L(x^{k+1}V) &= x^{k+1}(V) + \sum_{i=1}^{k+1} \left[\binom{k}{i} + \binom{k}{i-1}\right] x^{k+1-i}L^i(V) \\ &= x^{k+1}(V) + \sum_{i=1}^{k+1} \binom{k+1}{i} x^{k+1-i}L^i(V) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} x^{k+1-i}L^i(V). \end{aligned}$$

LEMMA 4. $L^{-1}(xV)=(x-L^{-1}L')L^{-1}(V)$.

Proof. $xV=xLL^{-1}(V)=xLL^{-1}(V)+L'L^{-1}(V)-L'L^{-1}(V)$. Applying Theorem 1 to the first two terms we get $xV=L(xL^{-1}(V))-L'L^{-1}(V)=L(xL^{-1}(V))-LL^{-1}L'L^{-1}(V)$. Hence $L^{-1}(xV)=xL^{-1}(V)-L^{-1}L'L^{-1}(V)=(x-L^{-1}L')L^{-1}(V)$.

THEOREM 2. $L^{-1}(x^nV)=(x-L^{-1}L')^nL^{-1}(V)$.

Proof. The case for $n=1$ is obvious from the preceding theorem. Assuming the validity for $n=k$, we get $L^{-1}(x^{k+1}V)=L^{-1}(xx^kV)=(x-L^{-1}L')L^{-1}(x^kV)=(x-L^{-1}L')(x-L^{-1}L')^kL^{-1}(V)=(x-L^{-1}L')^{k+1}L^{-1}(V)$.

HOW MUCH REDUNDANCY?

F. MAX STEIN, Colorado State University

Those of us who teach mathematics have the habit of figuratively and literally throwing up our hands when we hear a student say something like—a round circle. We have been trained, and we try to pass on to our students, to speak and write succinctly and avoid using more words than are absolutely necessary, at least in mathematical descriptions.

With my tongue in my cheek, I would like to present the following elementary lecture to illustrate a rather extreme case of redundancy.

Professor I. Q. Muddle shuffled into room 222 on the second floor of the Mathematics Building to meet his morning ten o'clock class. Although some of and, hence, the lemma is proved.

THEOREM 1.

$$L(x^nV) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} L^i(V).$$

Proof. The theorem is valid for $n=1$. Assuming the validity for $n=k$, we get

$$\begin{aligned} L(x^{k+1}V) &= xL(x^kV) + L'(x^kV) \\ &= \sum_{i=0}^{k} \binom{k}{i} x^{k+1-i} L^i(V) + \sum_{i=0}^{k} \binom{k}{i} x^{k-i} L^{i+1}(V). \end{aligned}$$

Noting that $\binom{k}{k+1} x^0 L^{k+1}(V)=0$, we obtain the following by isolating the $i=0$ term from the first summation and shifting the index in the second summation:

$$\begin{aligned} L(x^{k+1}V) &= x^{k+1}(V) + \sum_{i=1}^{k+1} \left[\binom{k}{i} + \binom{k}{i-1}\right] x^{k+1-i} L^i(V) \\ &= x^{k+1}(V) + \sum_{i=1}^{k+1} \binom{k+1}{i} x^{k+1-i} L^i(V) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} x^{k+1-i} L^i(V). \end{aligned}$$

LEMMA 4. $L^{-1}(xV)=(x-L^{-1}L')L^{-1}(V)$.

Proof. $xV=xLL^{-1}(V)=xLL^{-1}(V)+L'L^{-1}(V)-L'L^{-1}(V)$. Applying Theorem 1 to the first two terms we get $xV=L(xL^{-1}(V))-L'L^{-1}(V)=L(xL^{-1}(V))-LL^{-1}L'L^{-1}(V)$. Hence $L^{-1}(xV)=xL^{-1}(V)-L^{-1}L'L^{-1}(V)=(x-L^{-1}L')L^{-1}(V)$.

THEOREM 2. $L^{-1}(x^nV)=(x-L^{-1}L')^nL^{-1}(V)$.

Proof. The case for $n=1$ is obvious from the preceding theorem. Assuming the validity for $n=k$, we get $L^{-1}(x^{k+1}V)=L^{-1}(xx^kV)=(x-L^{-1}L')L^{-1}(x^kV)=(x-L^{-1}L')(x-L^{-1}L')^kL^{-1}(V)=(x-L^{-1}L')^{k+1}L^{-1}(V)$.

HOW MUCH REDUNDANCY?

F. MAX STEIN, Colorado State University

Those of us who teach mathematics have the habit of figuratively and literally throwing up our hands when we hear a student say something like—a round circle. We have been trained, and we try to pass on to our students, to speak and write succinctly and avoid using more words than are absolutely necessary, at least in mathematical descriptions.

With my tongue in my cheek, I would like to present the following elementary lecture to illustrate a rather extreme case of redundancy.

Professor I. Q. Muddle shuffled into room 222 on the second floor of the Mathematics Building to meet his morning ten o'clock class. Although some of his colleagues had mentioned to him that he had been tending more and more to needless repetition in his teaching and that too much was enough, his only comment was—"I merely say over again those things I repeat."

"Students and pupils," he started, "today we shall discuss the intersection of a curve, bounded above and below and on each side and remaining finite, with a curve that is unbounded and goes to infinity. In particular we shall consider a round circle with center C in the middle. We shall then draw a diameter through the center from one side to the other using a portion of a straight line of infinite length and with zero curvature. This line segment intersects the circle in two finite real points on opposite sides of the circle, say A and B.

"If we now circumscribe the circle on the outside with a rectangular square with four 90° right angles by letting two of the sides of the square touch the circle tangentially at A and B, we can obtain an approximation to the area enclosed by the circle by determining the area enclosed by the square.

"Since the length of the diameter d, the distance across a circle, is twice as long as the distance from the center to the circumference, the length of the radius r, then the distance along one of the side edges of the square is $2r$. The area enclosed by a square is known to be the square of a side of the square, so the area in this case is $4r^2$, a rough approximation, that is not too accurate, to the area enclosed by the circle.

"If on the other hand we had inscribed a regular square inside the circle with two of its corner vertices at A and B, the area enclosed by the inscribed square could readily be determined correctly to be exactly $2r^2$ by dividing the square into four congruent three-sided triangles of the same dimensions. We thus obtain another rough approximate value of the area enclosed by a circle.

"We might think that the exact true area enclosed by a circle would perhaps be half-way between these two results, or $3r^2$. However, a nearly exact approximation, for some purposes, is $22r^2/7$, while the correct true value is πr^2.

"With the preceding result at our disposal, we now consider a related problem that uses this result. We draw two concentric circles with centers at the same point. The circular annulus between these two circles is then bisected in two by a common diameter to each of them into two halves of the same size. It can be seen visually that they are of the same size if one half is folded over onto the other. It is readily determined that the area enclosed between the two circles is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$, where R is the radius of the larger circle and r is the radius of the smaller one, obtained by taking the difference by subtraction between the two areas.

"Our time is up for today, but tomorrow I have a triple of three topics from which I'll choose my lecture:

(1) Polynomials with a finite number of terms,

(2) Common logarithms to the base 10, or

(3) Curves that approach asymptotes in an asymptotic manner.

The class is dismissed."

A DIFFERENCE METHOD FOR OBTAINING Z-TRANSFORMS

DANIEL C. FIELDER, Georgia Institute of Technology

1. Introduction. Many recent texts and papers [1]–[6] on Z-Transforms and related subjects use a power series approach to introduce the direct Z-Transforms. While certain advantages accrue from this method, the search for a closed form of a power series in $(1/z)$ is often tedious and sometimes difficult. A difference method for obtaining closed forms is discussed herein.

2. Difference approach. In order to avoid entirely the problem of series closure, a difference method similar to that of Gardner and Barnes [7] can be used to great advantage. The difference technique was originally developed for use with "jump functions." A jump function is a series of contiguous constants width pulses, the heights of which represent a function which has a value only at integral values of the independent variable.

In the Z-Transform approach, the function is sampled every T seconds. Although the result of sampling is actually a series of trapezoidal shaped pulses, the duration of each pulse is so short that for mathematical analysis the pulses can be replaced by a series of impulses whose impulsive strengths are equal to the areas of the pulses they replace. If the Dirac δ-function [8] is used to represent a unit impulse, the series form of the function $f^*(t)$ obtained by sampling $f(t)$ is

$$f^*(t) = \sum_{n=0}^{\infty} f(nT)\delta(t - nT). \tag{1}$$

A comparison of the Laplace Transforms of jump functions and the Z-Transforms of impulse-sampled functions indicates a marked mathematical similarity.

Consider the function $f^*(t)$. The Laplace Transform is $\mathcal{L}[f^*(t)] = F^*(s)$, and the Z-Transform is $Z[f^*(t)] = Z[f(t)] = F(z)$. $F(z)$ results from replacing e^{sT} in $F^*(s)$ by z. The first forward difference [9] of $f^*(t)$ is

$$\Delta f^*(t) = f^*(t + T) - f^*(t), \tag{2}$$

and its Laplace Transform is

$$\mathcal{L}[\Delta f^*(t)] = \int_0^\infty f^*(t + T)e^{-st}dt - F^*(s). \tag{3}$$

If t is replaced by $(\tau - T)$ in (3) and

$$e^{Ts}\int_0^T f^*(\tau)e^{-s\tau}d\tau \tag{4}$$

is added and subtracted to the right side of (3), the result is

$$\mathcal{L}[\Delta f^*(t)] = e^{Ts}\int_0^\infty f^*(\tau)e^{-s\tau}d\tau - F^*(s) - e^{Ts}\int_0^T f^*(\tau)e^{-s\tau}d\tau. \tag{5}$$

Since $f^*(\tau)=f(0)\delta(\tau)$ between 0 and T, the integral on the right in (5) is reduced to $e^{Ts}f(0)$. Thus,

$$\mathcal{L}[\Delta f^*(t)] = (e^{Ts} - 1)F^*(s) - e^{Ts}f(0), \tag{6}$$

and the corresponding Z-Transform becomes

$$Z[\Delta f^*(t)] = Z[\Delta f(t)] = (z - 1)F(z) - zf(0). \tag{7}$$

Frequently, $\Delta f^*(t)$ is simpler than $f^*(t)$. If the Z-Transform of the simpler function, i.e., $Z[\Delta f^*(t)]$, can be found by some means, it can be compared with the formal expression (7). A slight algebraic manipulation yields $F(z)$.

As an example, consider $f(t)=c$, or $f^*(t)=c^*$. The first difference is zero, and $Z[\Delta c^*]=0$. Since $f(0)$ is c, (7) immediately yields

$$F(z) = Z[c^*] = (zc)/(z - 1) \cdot \tag{8}$$

As a second example, consider $f^*(t)=t^*$. The first difference is T^*, which has, according to (6), the Z-Transform

$$Z[T^*] = (zT)/(z - 1) \cdot \tag{9}$$

Application of (7) yields

$$F(z) = Z[t^*] = \frac{zT}{(z-1)^2} \cdot \tag{10}$$

The extension to higher order differences is readily made by means of

$$\begin{aligned} Z[\Delta^n f^*(t)] &= (z - 1)^n F(z) - z(z - 1)^{n-1}f(0) - z(z - 1)^{n-2}\Delta f(0) \\ &\quad - z(z - 1)^{n-3}\Delta^2 f(0) - \cdots - z(z - 1)\Delta^{n-2}f(0) - z\Delta^{n-1}f(0), \end{aligned} \tag{11}$$

where $\Delta^k f(0)$ is the kth difference of $f(t)$ evaluated at $t=0$. (Although Gardner and Barnes do not directly use higher order differences for this purpose, the application of higher order differences to their jump functions is obvious.) A complicated function often can be differenced n times until a function having an obvious Z-Transform is found. Use of (11) leads to the appropriate $F(z)$. It is interesting to note that the expression (11) will yield correct results even if $f^*(t)$ is grossly overdifferenced. A final example follows. Consider $f(t)=t^2$ or $f^*(t)=(t^2)^*$. The differences are calculated to be

$$\Delta(t^2)^* = (2tT + T^2)^*; \qquad \Delta f(0) = T^2, \qquad \Delta^2(t^2)^* = (2T^2)^*. \tag{12}$$

Application of (9) and (11) yield

$$Z[\Delta^2(t^2)^*] = \frac{2zT^2}{(z-1)} = (z - 1)^2F(z) - z(z - 1)0 - zT^2, \tag{13}$$

from which it follows that

$$F(z) = \frac{T^2 z(z + 1)}{(z-1)^3} \cdot \tag{14}$$

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9. K. S. Kunz, Numerical analysis, McGraw-Hill, New York, 1957, 38–41.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent directly to Dmitri Thoro, San Jose State College, San Jose, California 95114.

150 Puzzles in Crypt-Arithmetic. By Maxey Brooke. Dover, New York, 1963. 72 pp., paperbound $1.00.

Here, for the first time, we have a book devoted entirely to cryptarithms' alphametics, and other puzzles of their general type. With the great spread of interest in such problems in recent years this diversified selection of teasers will be welcomed by the tens of thousands of enthusiasts who enjoy this special pastime in recreational mathematics.

Puzzles of this type are familiar to readers of this MAGAZINE. Their solutions rarely require more than a very sound knowledge and understanding of the basic facts and operations of simple arithmetic, coupled with sound reasoning and often some patience: probably the main reason for their popularity.

In selecting problems for this little volume, Maxey Brooke has ranged widely throughout the whole field of such puzzles. He includes old classics, some already well known, and leads through the early cryptarithms right up to modern alphametics. And, in addition to a complete listing of answers, he outlines a few detailed solutions with general observations that will greatly help newcomers to this fascinating pastime.

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The purist may be irritated occasionally by frequent use of the word "number" where "digit" was meant.

In Puzzle No. 51, we may be surprised to find the *exact* value of π suggested as 41056/13069; i.e., $3.1417 \cdots$. This anachronism could have been avoided by use of the abbreviation "(*approx.*)"

Some pairs of puzzles are identical! For example, 58 and 112, 79 and 111, 94 and 133, 99 and 137; also No. 69 duplicates the puzzle that is given and discussed on pages 8 and 9.

In Nos. 94 and 133, and more obviously in No. 24, there is the very clear suggestion that unity is a prime number. The 1-digit primes, of course, are 2, 3, 5, 7: unity is very definitely not a prime.

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J. A. H. HUNTER, Toronto, Ontario

A Long Way from Euclid. By Constance Reid, Thomas Y. Crowell Co., New York, 1963. ix+292 pp., $5.00.

This book, by the author's own admission, was based in part on another one she had produced previously. So for those who have read her *Introduction to Higher Mathematics*, there will be occasional repetition. Yet this does not detract from the merits of each.

Neither book makes demands on the reader beyond some remote contact with elementary plane geometry, and each is written principally for the so-called general reader. Where the *Introduction* was essentially a sample showcase to whet the appetite, a *Long Way from Euclid* picks as a theme some typical knotty problems facing mathematicians from time to time and shows how the quest for solution brought forth new discoveries.

The book starts out with some preliminary historical background of the Pythagoreans and culminates with a glimpse into decision theory with a discussion of Tarski's elementary plane geometry E_2. Intermediate topics include the four square and the three square theorems, primality, and countability.

There may be argument with regards to the concept of a function set forth in this volume; the rule and the function seem to be inseparable. This reviewer chooses to consider these as separate entities, and that the function may well exist without an explicit rule for the mapping. These points may be too subtle to present at this level; yet I would be happier if some incidental comment were included to that effect.

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W. G. CHINN, San Francisco Unified School District

Enrichment Mathematics for High Schools. Twenty-eighth Yearbook, National Council of Teachers of Mathematics, Washington, D. C., 1963. ix+388 pp. $1.50 (paper), $3.00 (hard cover).

The principal objective of this yearbook is to present content-material for use as enrichment material for talented high school students, especially for those who plan to go to college. The book is divided into two sections: Section I: *The High School Years*, (17 articles); and Section II: *The Transition to College*, (10 articles).

The book gives the teacher ample advice and instruction as to how to integrate the various units of study presented into the actual teaching situation. It makes clear, for example, that some articles could be given to individual students for self study, while others should be joint projects involving both the teacher and the student, a mutual voyage of discovery, so to speak. Other combinations of student and teacher study relationships are suggested, and at various grade levels at that.

The reader who leans to the practical or applied side of mathematics will not be disappointed with this book, and will reap equal benefit along with those who wish to learn more about the structure of mathematics. The article on "Computing a Lunar Eclipse," by Samuel L. Greitzer is a good example of mathematics in action at the elementary level.

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The twenty-seven articles are written by top-notch mathematicians, teachers, and expositors, and range over the subject areas of number theory, geometry, algebra, group theory, probability, linear programming, matrix theory, knot theory, and many other fields of study.

Every reader will find in this text one or more articles which he particularly enjoys. Those interested in number theory will be impressed by the article, "Recent Information on Primes," by Paul C. Rosenbloom which explains how certain large numbers of the form $n=(k\cdot 2^m)+1$ are tested for primeness, a very hard question for any teacher to answer without help of this sort. A later article by Waclaw Sierpinski, "Some Unsolved Problems of Arithmetic," contains one of the best lists of unsolved problems in number theory to be found anywhere.

Up-to-date subjects are well represented, such as John C. Kemeny's excellent article on "Random Walks," and the one by R. H. Crowell on, "Knots and Wheels." On the other hand, to create a nice balance, there are articles on classical techniques in analysis, such as the article, extremely well done, on "Generating Functions," by Nathan J. Fine. The reviewer regrets there is not space to mention and praise each article in this yearbook.

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C. D. OLDS, San Jose State College

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The distinguished Professor Wall obviously has some axes he wishes to grind· While some of the aims he expresses are indeed laudable, I fear his book suffers either from a lack of clear definition of its aim or from a failure, in my judgment, to carry out the author's plan in an *effective* way.

The dust jacket, for which Professor Wall may well not be responsible, claims: "The book is self-sufficient and assumes no mathematical background on the part of the reader. In fact, the intelligent high school student will find himself capable of meeting the mathematical challenges presented here." I doubt this very much. The Preface contains the following remarks. "This book is intended to lead the student to develop his mathematical ability, to help him learn the art of mathematics, to teach him to create mathematical ideas. This is not a compendium of mathematical facts and inventions to be read over as a connoisseur of art looks over the paintings in a gallery. It is, instead, a sketchbook in which the reader may try his hand at mathematical discovery." "This book is written for the person who seeks an intellectual challenge and who can find genuine pleasure in spending hours and even weeks in constructing proofs for the theorems of one chapter or even a portion of one chapter. It is a book which may be useful for the formal student and which is also intended for the person who is not in school but who wants to study mathematics on his own. A person who has worked through this book can be regarded as a good mathematician." "Much mathematical jargon, to some of which any but the most unspoiled person has been exposed, has very little real meaning." A little later we read, "The mathematician is an artist whose medium is the mind and whose creations are ideas."

I will admit that part of my lack of much enthusiasm for this book can be attributed to a difference in tastes—and *de gustibus non est disputandum*. For example, I prefer $\int_a^b x dx$ to $\int_a^b I$. (Creative mathematicians do use mnemonic techniques.) (Professor Wall introduces $\int_a^b H$ on page 15. Much later, in discussing a space $C_m[a, b]$, he does use $\int_a^b \phi_p dX$.) I prefer to use the word *function* and define that concept clearly. (Professor Wall insists on the term "simple graph," tries manfully not ever to mention "function." His extensive glossary does not contain "function," though the word did creep into the text.) I prefer l.u.b. M to Professor Wall's choice of terms when he writes (page 8), "Axiom XVIII. If M is a set each element of which is a number and if there is a number which is less than no number of M, then there is a number k which is less than no number of M such that, if k' is a number less than k, then k' is less than

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The dust jacket, for which Professor Wall may well not be responsible, claims: "The book is self-sufficient and assumes no mathematical background on the part of the reader. In fact, the intelligent high school student will find himself capable of meeting the mathematical challenges presented here." I doubt this very much. The Preface contains the following remarks. "This book is intended to lead the student to develop his mathematical ability, to help him learn the art of mathematics, to teach him to create mathematical ideas. This is not a compendium of mathematical facts and inventions to be read over as a connoisseur of art looks over the paintings in a gallery. It is, instead, a sketchbook in which the reader may try his hand at mathematical discovery." "This book is written for the person who seeks an intellectual challenge and who can find genuine pleasure in spending hours and even weeks in constructing proofs for the theorems of one chapter or even a portion of one chapter. It is a book which may be useful for the formal student and which is also intended for the person who is not in school but who wants to study mathematics on his own. A person who has worked through this book can be regarded as a good mathematician." "Much mathematical jargon, to some of which any but the most unspoiled person has been exposed, has very little real meaning." A little later we read, "The mathematician is an artist whose medium is the mind and whose creations are ideas."

I will admit that part of my lack of much enthusiasm for this book can be attributed to a difference in tastes—and *de gustibus non est disputandum*. For example, I prefer $\int_a^b x dx$ to $\int_a^b I$. (Creative mathematicians do use mnemonic techniques.) (Professor Wall introduces $\int_a^b H$ on page 15. Much later, in discussing a space $C_m[a, b]$, he does use $\int_a^b \phi_p dX$.) I prefer to use the word *function* and define that concept clearly. (Professor Wall insists on the term "simple graph," tries manfully not ever to mention "function." His extensive glossary does not contain "function," though the word did creep into the text.) I prefer l.u.b. M to Professor Wall's choice of terms when he writes (page 8), "Axiom XVIII. If M is a set each element of which is a number and if there is a number which is less than no number of M, then there is a number k which is less than no number of M such that, if k' is a number less than k, then k' is less than *some* number of M. The number k is called *the least number which no number in M exceeds.*" I prefer to *introduce novices* to $\log_a x$ in a motivated manner and in a form different from $L(x)/L(a)$ where L is a certain simple graph and $L(x)$ is a certain area. But "motivation" is absent from this book, apparently by design. (Creative mathematicians do like motivations.) The book is devoid of historical references—and I can't seem to purge myself of a liking for such, in moderation. I don't think I could, for example, pointedly avoid calling a theorem "the Bolzano-Weierstrass theorem" for long. I prefer never to use $a=0.82$ when I mean only "a has the approximate value 0.82." I like to use "if and only if." (Languages do have meaningful idioms.) I dislike the notation $[ab; cd]$ for a certain rectangular point set; I wouldn't know how to read $[142; 2367]$.

Ah, but the book is worth reading. But if you follow the ground rules, *i.e.*, study it "on your own," you may not appreciate some of its truly fine discussions as much as a reader who is not in his solitary novitiate.

L. H. LANGE, San Jose State College

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

PROPOSALS

565. *Proposed by Maxey Brooke, Sweeny, Texas.*

Henry, Hyram and Hyman inherited a circular farm. It is not enough to divide it into three equal areas. In order that they can share equally the fencing costs, the three portions must also have equal perimeters. Can you help the heirs?

566. *Proposed by Martin J. Cohen, Beverly Hills, California.*

Prove that n is a square-free integer (i.e., the product of distinct primes) if and only if

$$\sum_{d|n} \phi(d)\sigma(d^{k-1}) = n^k$$

for all integers $k \geqq 2$.

567. *Proposed by L. Carlitz, Duke University.*

Points A_1, A_2 are marked on the side BC of the triangle ABC so that $BA_1 = A_1A_2 = A_2C$, similarly B_1, B_2 on CA and C_1, C_2 on AB. Let A' be the point of *some* number of M. The number k is called *the least number which no number in* M *exceeds*." I prefer to *introduce novices* to $\log_a x$ in a motivated manner and in a form different from $L(x)/L(a)$ where L is a certain simple graph and $L(x)$ is a certain area. But "motivation" is absent from this book, apparently by design. (Creative mathematicians do like motivations.) The book is devoid of historical references—and I can't seem to purge myself of a liking for such, in moderation. I don't think I could, for example, pointedly avoid calling a theorem "the Bolzano-Weierstrass theorem" for long. I prefer never to use $a=0.82$ when I mean only "a has the approximate value 0.82." I like to use "if and only if." (Languages do have meaningful idioms.) I dislike the notation $[ab; cd]$ for a certain rectangular point set; I wouldn't know how to read $[142; 2367]$.

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568. *Proposed by C. W. Trigg, San Diego, California.*

What is the nature of n if $\sum_{k=1}^{n} k^6$

is divisible by $\sum_{k=1}^{n} k^2$?

569. *Proposed by Frank Dapkus, Seton Hall University.*

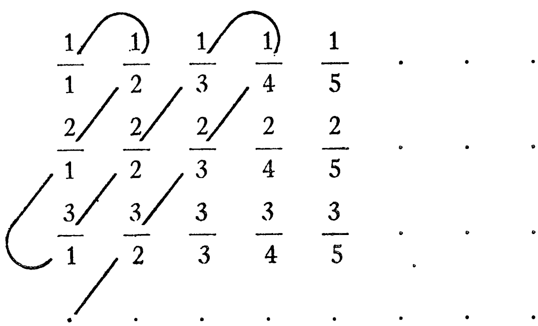
Is it true that of all the surfaces of revolution only the sphere and circular cylinder have the property that the areas of zones of equal thickness are equal?

570. *Proposed by Leon Bankoff, Los Angeles, California.*

A billiard ball is placed at a point P on a circular billiard table with center O and radius $R=1$, and is hit so that it returns to its starting point after rebounding from points A and B on the circumference. If PD is the altitude to side AB of the triangle PAB, determine the location of P so that O divides PD in the Golden Ratio $(\sqrt{5}-1)/2$. Neglect friction and spin of the ball, and assume perfect elasticity of the cushions.

571. *Proposed by Herta Taussig Freitag, Hollins College, Virginia.*

According to Cantor's "diagonal procedure," the denumerability of the rationals may be established by ordering them in the manner indicated below:



Thus,

$$\frac{1}{1} \leftrightarrow 1, \qquad \frac{1}{2} \leftrightarrow 2, \qquad \frac{2}{1} \leftrightarrow 3, \text{ etc.}$$

Design a matching formula between any given fraction a/b and the corresponding natural number n.

SOLUTIONS

Late Solutions

545, 546, 547, 548, 550. *Stanton Philipp, Seal Beach, California.*

Quadratic Residue

541. [January, 1964] *Proposed by J. Barry Love, Eastern Baptist College, Pennsylvania.*

Let p be a prime, and let n be the smallest positive quadratic residue (mod p). Show that $n<1/2+\sqrt{p}$.

Solution by Josef Andersson, Vaxholm, Sweden. (Translated and paraphrased by the editor.)

It is evident that the "smallest quadratic residue (mod p)" is $1 < 1/2 + \sqrt{p}$. The proposer probably meant to consider the "second smallest quadratic residue." Also suppose that $p>5$ because $4 \nless 1/2+\sqrt{5}$.

With these restrictions, we find three cases:

I. $n=2$, $p\geqq 7$, then $2<1/2+\sqrt{p}$
II. $n=3$, $p\geqq 11$, then $3<1/2+\sqrt{p}$
III. $n=4$, $p>13$ (in fact $\geqq 19$), then $4<1/2+\sqrt{p}$.

The revised theorem is true. If one considers the parity of the indices of 2, 3 from a table one can avoid the calculations.

A Conditional Alphametic

544. [March, 1964] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm (alphametic)

$$ONE+TWO+SIX=NINE$$

in the base 10, with the following conditions:

a) $ONE<TWO<SIX$
b) $2|TWO$, $6|SIX$, $9|NINE$ where $a|b$ means "a divides b."

Solution by Sister Mary Joy, Notre Dame College, St. Louis, Missouri.

Since each letter represents a different digit, it can readily be seen from condition (a) that $O<T<S$, $S\geqq O+2$, $T\geqq O+1$, and from condition (b) that TWO and SIX are both even.

Observe that E occupies the unit's place in the sum. Thus, $O+X$ must be 10. Both X and O are single digits, neither can be zero, nor can the sum be greater than $1E$. From the fact that N occupies the ten's place in the sum, it follows that $W+I=9$. Also, $W+I\neq 10$ as there is 1 ten carried from the unit's column.

Thus there are four possible ordered pairs for X and O: (8,2), (2,8), (6,4) and (4,6). Now pairs of addends for W and I are chosen such that neither addend duplicates a digit already taken. Possibilities for S are then chosen such that $6|SIX$, where $S\geqq O+2$. If no duplication has occurred thus far, T is chosen so that $O+T+S=NI$. With still no duplication of digits, E is determined such that $9|NINE$.

Consequently the solution is found to be

ONE	217
TWO	392
SIX	408
NINE	1017

Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Maxey Brooke, Sweeny, Texas; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; David M. Cohen, East Midwood Day School, Brooklyn, New York; Martin J. Cohen, Beverly Hills, California; Michael P. Cozmanoff, Lew Wallace High School, Gary, Indiana; John A. Dossy, Illinois State University, Normal, Illinois; Joseph M. Fine, Massachusetts Institute of Technology; C. E. Franti, Berkeley, California; Philip Fung, Fenn College, Ohio; Harry M. Gehman, SUNY at Buffalo, New York; Murray Geller, Jet Propulsion Laboratory, Pasadena, California; Anton Glasser, Pennsylvania State University, Abington, Pennsylvania; Garold F. Gregory, Forest Disease Research Laboratory, Delaware, Ohio; C. T. Haskell, California State Polytechnic College, San Luis Obispo, California; Burton S. Holland, Harpur College, New York; William R. Holt, Delaware, Ohio; J. A. H. Hunter, Toronto, Ontario, Canada; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Janice Langan, Lew Wallace High School, Gary, Indiana; John W. Milsom, Texas A and I, Kingsville, Texas; Wa Hin Ng, San Francisco, California; C. C. Rice, IBM, Endicott, New York; Perry A. Scheinok, Hahnemann Medical College, Philadelphia, Pennsylvania; C. W. Trigg, San Diego, California; A. M. Vaidya, Pennsylvania State University; J. S. Vigder, Ottawa, Canada; Thomas Wojtan, Lew Wallace High School, Gary, Indiana; Dale Woods, Northeast Missouri State TeachersCollege; Charles Ziegenfus, Madison College, Virginia; and the proposer.

A Parameter

545. [March, 1964] *Proposed by C. Stanley Ogilvy, Hamilton College, New York.*

A curve is given by the parametric equations $x=2t/(1+t^2)$, $y=(1-t^2)/(1+t^2)$. What is the geometric meaning of the parameter t?

Solution by R. T. Coffman, Richland, Washington.

By squaring and adding the given equations, we obtain $x^2+y^2=1$, from which $x=\cos\theta$, $y=\sin\theta$, where

$$\theta = \arc\tan\frac{y}{x}\cdot$$

Also,

$$\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\frac{1+t^2}{2t}+\frac{1-t^2}{2t}=\frac{1}{t}\cdot$$

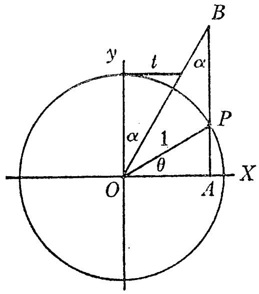
Therefore,

$$t=\frac{\cos\theta}{1+\sin\theta}\cdot$$

In the figure, $OA=\cos\theta$ and $AB=1+\sin\theta$. Also,

$$\frac{\cos\theta}{1+\sin\theta}=\tan\alpha=t.$$

The tangent drawn from (0, 1) to intersect OB has a length t since geometrically its length is $\tan\alpha$. There are a number of other possible representations of t, such as $t=\tan(45°-\theta/2)$.



Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Frank Dapkus, Seton Hall University; Charles Franti, Berkeley, California; Philip Fung, Fenn College, Ohio; Harold A. Heckart, Illinois College, Jacksonville, Illinois; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Margaret Lindsey, University of Alabama; Perry A. Scheinok, Hahnemann Medical College, Philadelphia, Pennsylvania; John Wessner, Melbourne High School, Melbourne, Florida; and the proposer.

A Solution in Integers

546. [March, 1964] *Proposed by D. Rameswar Rao, Secunderabad, India.*

Solve in integers the equation $x^2-y^2=X^6-Y^6$.

Solution by J. A. H. Hunter, Toronto, Ontario, Canada.

Setting $X=(ab+cd)/2$, $Y=(ab-cd)/2$, where ab and cd are both even or both odd, and substituting, we have:

$$(x+y)(x-y) = abcd(3a^2b^2+c^2d^2)(a^2b^2+3c^2d^2)/16$$

so

$$x+y = ac(3a^2b^2+c^2d^2)/4$$
$$x-y = bd(a^2b^2+3c^2d^2)/4$$

whence

$$\begin{cases} x = \pm\,(3a^3b^2c+a^2b^3d+ac^3d^2+3bc^2d^3)k^6/8 \\ y = \pm\,(3a^3b^2c-a^2b^3d+ac^3d^2-3bc^2d^3)k^6/8 \\ X = \pm\,(ab+cd)k^2/2 \\ Y = \pm\,(ab-cd)k^2/2 \end{cases}$$

the coordinating constant k having been introduced for complete generality, all subject to ab and cd both even or both odd. The minimal solution,

$$(x, y; X, Y) = (8, 1; 2, 1)$$

is derived with $a=c=d=1$, $b=3$, $k=1$.

Also solved by Josef Andersson, Vaxholm, Sweden; Merrill Barneby, University of North Dakota; Martin J. Cohen, Beverly Hills, California; J. M. Fine, Massachusetts Institute of Technology (partial solution); David A. Klarner, University of Alberta, Canada; J. S. Vigder, Ottawa, Canada; and the proposer.

A Mean Inequality

547. [March, 1964] *Proposed by Daniel I. A. Cohen, Midwood High School, Brooklyn, New York.*

Prove that the arithmetic mean of the nth powers of a set of numbers is never less than the nth power of the arithmetic mean of the numbers.

Solution by A. M. Vaidya, Pennsylvania State University.

This is a well-known result and proofs can be found in any sufficiently old book on Algebra, such as that of Chrystal (Vol. 2, page 48). However, the following simple proof does not seem to be in print.

We use the fact that for $n \geqq 1$, the curve $y = x^n$ is an increasing, convex curve for which the chord is never below the curve. On this curve, consider the convex polygon formed by the points (a_i, a_i^n), $i = 1, 2, \cdots, k$. The center of gravity

$$G\left(\frac{\sum a_i}{k}, \frac{\sum a_i^n}{k}\right)$$

of this polygon is within the polygon and so is not below the curve. The point

$$H\left(\frac{\sum a_i}{k}, \left(\frac{\sum a_i}{k}\right)^n\right)$$

is on the same ordinate as G, but it is on the curve. Therefore we must have

$$\frac{\sum a_i^n}{k} \geqq \left(\frac{\sum a_i}{k}\right)^n,$$

as was to be proved.

(It can be similarly proved that for $n \leqq 1$, the inequality has to be reversed.)

Also solved by Josef Andersson, Vaxholm, Sweden; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Martin J. Cohen, Beverly Hills, California; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; and the proposer.

Brown and *Konhauser* located the problem in "The USSR Olympiad Book," published in 1962.

Exponential Inequality

548. [March, 1964] *Proposed by Leo Moser, University of Alberta.*

Show that if a and c are positive reals and b and d positive integers with $b \geqq d$, then $(a-1)b \geqq (c-1)d$ implies $a^b \geqq c^d$.

Solution by H. W. Gould, West Virginia University.

A generalized form of the Bernoulli inequality comes to mind.

THEOREM. *Let* $x \geqq -1$. *Then* $(1+x)^\alpha \leqq 1 + \alpha x$ *if* $0 < \alpha < 1$, *and* $(1+x)^\alpha \geqq 1 + \alpha x$ *if* $\alpha < 0$ *or* $\alpha > 1$. *Equality holds in each case when* $x = 0$.

A detailed proof of this theorem is given in Korovkin's pamphlet on inequalities (pp. 19–22 of the Russian edition), and many other sources could be cited. For example, the essence of the theorem is on pp. 143–144 of Hardy's *Course of Pure Mathematics*, 9th edition, 1944.

The present problem may be solved by use of *either* part of the general theorem.

On the one hand,

$$(a - 1)b \geqq (c - 1)d \quad \text{so that} \quad a - 1 \geqq (c - 1)d/b$$

By the *first* part of the theorem (with $x = c-1$, $\alpha = d/b$)

$$(c - 1)d/b \geqq c^{d/b} - 1, \quad \text{since} \quad b \geqq d,$$

and so $a-1 \geqq c^{d/b}-1$, or $a \geqq c^{d/b}$ which gives the desired result $a^b \geqq c^d$.

On the other hand, $(c-1)d \leqq (a-1)b$ so that

$$c - 1 \leqq (a - 1)b/d \leqq a^{b/d} - 1$$

by the *second* part of the theorem. Hence again

$$c \leqq a^{b/d} \quad \text{or} \quad c^d \leqq a^b.$$

Also solved by Josef Andersson, Vaxholm, Sweden; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Dale Woods, Northeast Missouri State Teachers College; and the proposer.

Clairaut Equation

549. [March, 1964] *Proposed by Murray S. Klamkin, SUNY at Buffalo, New York.*

The solution of the Clairaut equation $y = xy' + F(y')$ is obtained by setting $y' = c$ which gives $y = cx + F(c)$. Determine the most general first order differential equation in which the solution can be obtained in this manner.

Solution by Josef Andersson, Vaxholm, Sweden. (*Translated by the editor.*)

If the equation is written $y = \Phi(x, y')$ it follows that

$$y' = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y'} \cdot y''.$$

The solution $y' = C$, $y'' = 0$ gives

$$\frac{\partial \Phi}{\partial x} = y'$$

and $\Phi = xy' + F(y')$.

The Clairaut equation is therefore unique.

Also solved by the proposer.

Consecutive Square Sums

550. [March, 1964] *Proposed by Brother U. Alfred, St. Mary's College, California.*

The following is a set of equations in which n consecutive integers and $n+1$ consecutive integers have equal sums of squares.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

a) What would be the first term on the left for the case of n and $n-1$ consecutive integers?
b) Prove that your result holds in the general case.

Solution by Monte Dernham, San Francisco, California.

(a) We observe that the sequence 3, 10, 21 consists of the second, fourth and sixth triangular numbers. Hence it appears that the first left-hand term for the case of n and $n-1$ consecutive integers would be the square of the $2(n-1)$st triangular number. That is:

$$(2n^2 - 3n + 1)^2.$$

(b) The corresponding general equation would then be

$$(2n^2 - 3n + 1)^2 + (2n^2 - 3n + 2)^2 + \cdots + (2n^2 - 2n - 1)^2 + (2n^2 - 2n)^2$$
$$= (2n^2 - 2n + 1)^2 + (2n^2 - 2n + 2)^2 + \cdots + (2n^2 - n - 1)^2.$$

We proceed to prove that this equation is an identity, and thus that the foregoing result holds in the general case.

Transpose to the right side of the equation all terms on the left, except the first, pairing the original first right-hand term with the last of the former left-hand terms, the second with the former second to the last, and so on, thus:

$$\begin{aligned}(2n^2 - 3n + 1)^2 &= ([2n^2 - 2n + 1]^2 - [2n^2 - 2n]^2) \\ &\quad + ([2n^2 - 2n + 2]^2 - [2n^2 - 2n - 1]^2) \\ &\quad + [(2n^2 - 2n + 3]^2 - [2n^2 - 2n - 2]^2) + \cdots \\ &\quad + ([2n^2 - n - 1]^2 - [2n^2 - 3n + 2]^2) \\ &= (4n^2 - 4n + 1)(1 + 3 + 5 + \cdots + [2n - 3]) \\ &= (2n - 1)^2(n - 1)^2 = (2n^2 - 3n + 1)^2\end{aligned}$$

which completes the proof.

Also solved by Josef Andersson, Vaxholm, Sweden; L. Carlitz, Duke University; C. E. Franti, Berkeley, California; Herta T. Freitag, Hollins College, Virginia; Edwin V. Gadecki, Northeastern University, Massachusetts; Murray Geller, Jet Propulsion Laboratory, Pasadena, California; J. A. H. Hunter, Toronto, Ontario, Canada; Roop N. Kesarwani, Wayne State University; Wa Hin Ng, San Francisco, California; Walter Penney, Greenbelt, Maryland; Wade E. Philpott, Lima, Ohio; Adrian Struyk, Paterson, New Jersey; C. W. Trigg, San Diego, California; J. S. Vigder, Ottawa, Canada; Charles Ziegenfus, Madison College, Virginia; and the proposer.

Trigg pointed out that this generalization of Pythagorean numbers appears in *Sphinx*, 7(April, 1937), page 72, and in *Scripta Mathematica*, 5(January, 1938), page 32, where the rule is given: To find $n+1$ consecutive squares the sum of which is equal to that of the n following squares, $2n(n+1)$ is the middle member of the set. Another pertinent reference is by H. L. Alder, "n and $n+1$ Consecutive Integers with Equal Sums of Squares," *American Mathematical Monthly*, 69 (April, 1962), pp. 282–5.

Comment on F21

F21. [January, 1964]. *Comment by James F. Ramaley, University of California, Berkeley.*

An attempt to evaluate the integral $\int dx/x \ln x$ by parts instead of the straightforward substitution leads to

$$\int \frac{dx}{x \ln x} = 1 + \int \frac{dx}{x \ln x}\cdot$$

One might conclude that $0=1$. However, such a conclusion results from confusing the "function" 1 and the "value of the function" 1. This is clearly shown when one inserts the limits of integration:

$$\int_a^b \frac{dx}{x \ln x} = 1\Big|_a^b + \int_a^b \frac{dx}{x \ln x}$$

whence there is obviously no paradox.

A similar example was submitted by *Herta T. Freitag, Hollins College, Virginia.*

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 344. If $\bar{r}$ denotes the mean distance between two random points in a sphere of radius r (with uniform distribution with respect to volume), show that $3r/2 > \bar{r} > 3r/4$.

[*Submitted by M. S. Klamkin and J. D. Newman*]

Q 345. Which is larger

$$7^{\sqrt{5}} \quad \text{or} \quad 5^{\sqrt{7}}?$$

[*Submitted by David L. Silverman*]

Q 346. The product of three consecutive even integers is 87 * * * * * 8. Find the integers and supply the missing digits in the product.

[*Submitted by C. W. Trigg*]

Q 347. Find the square of the number 1234.

[*Submitted by Robert A. Lee*]

Q 348. If three forces are in equilibrium they must be coplanar and concurrent.

[*Submitted by M. S. Klamkin*]

(Answers on page 321)

(16) $$\omega_p(p-3) < p-3.$$

However, it is not clear how to obtain an exact result like (14) for $\omega_p(p-3)$.

The congruence (15) can be generalized in an obvious way. Indeed if

$$(x)_{p-k} = \sum_{r=0}^{p-k-1} c_r^{(k)} x^{p-k-r},$$

then

$$c_r^{(k)} \equiv \frac{(-1)^r}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} (j+1)^{k+r} \pmod{p}.$$

In particular, it is clear from the above discussion, that there exist infinitely many pairs p, t with $t<p$ such that $\omega_p(t)<t$.

Supported in part by NSF grant GP-1593.

References

1. E. Lucas, Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier, Bull. Soc. Math. France, 6 (1878) 49–54.

2. J. Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.

ANSWERS

A 344. Let A' denote the image of A with respect to O, the center of the sphere. Then $OA+OB>AB$ (in general). Whence, $\overline{OA}+\overline{OB}>\overline{AB}$. But

$$\overline{OA} = \overline{OB} = \int_0^r r\cdot 4\pi r^2 dr \div \int_0^r 4\pi r^2 dr = 3r/4$$

Also $\overline{AB}+\overline{AB'}>2\overline{OA}$. Since $\overline{AB}=\overline{AB'}$ the inequalities follows.

A 345. $(7^{\sqrt{5}})^{\sqrt{35}}=7^{5\sqrt{7}}=(16,807)^{\sqrt{7}}>(15,625)^{\sqrt{7}}=5^{6\sqrt{7}}=5^{\sqrt{252}}>5^{\sqrt{245}}=5^{7\sqrt{5}}=(5^{\sqrt{7}})^{\sqrt{35}}$. Hence $7^{\sqrt{5}}>5^{\sqrt{7}}$.

A 346. No one of the digits is zero. $(4)(6)(8)=192$, and $(2)(4)(6)=48$. $\sqrt[3]{87}=4.4+$ and $\sqrt[3]{88}=4.4+$. Therefore, the product is $(442)(444)(446)=87526608$.

A 347. If $(WXYZ\cdots U)$ are the digits of an integer, it can be shown that $(WXYZ\cdots U)^2=(W^2+X^2+Y^2+\cdots+U^2)+(2W)X+(2WX)Y+(2WXYZ)T+\cdots+(2WXY\cdots)U$. Thus $(1234)^2=(1000^2+200^2+30^2+4^2)+(2\cdot 1000)200+(2\cdot 1200)30+(2\cdot 1230)4=1,522,756$.

A 348. If two of the forces are skew, it would be possible to get a nonzero moment about an axis intersecting these two axes. Consequently, these two forces must lie in a plane and intersect (possibly at infinity). Then the third force (by moments) must lie in this plane and be concurrent to the other two.

(Quickies on page 360)